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CONVERGENCE RATE OF A GENERALIZED ADDITIVE SCHWARZ ALGORITHM [*1)]

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Abstract

The convergence rate of a generalized additive Schwarz algorithm for solving boundary value problems of elliptic partial differential equations is studied. A quantitative analysis of the convergence rate is given for the model Dirichlet problem. It will be shown that a greater acceleration of the algorithm can be obtained by choosing the parameter suitably. Some numerical tests are also presented in this paper.

Mathematics subject classification: 35J67, 65N55.
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1. Introduction

A classical mathematical approach, the Schwarz alternating algorithm (see, e.g., in [12]), appears to offer promise for the parallel solution of the very large systems of linear or nonlinear elliptic problems in elasticity, fluid dynamics or other important areas. Its advantage in parallelism, wide applicability and great flexibility in implementation make Schwarz algorithm a competitive technique in parallel computations. As a result, Schwarz algorithms have attracted much attention from researchers in the field of parallel computation as well as theoreticians. The early contributions relating to Schwarz algorithms can also be seen in [9, 10, 14]. Some recent progress in this field can be seen in [8, 11, 13] and the references therein.

A generalized additive Schwarz algorithm is presented in the paper. The approach uses robin condition on the inner boundaries of the subproblems. The use of Robin boundary condition as interfacial transmission conditions in domain decomposition was introduced by P. L. Lions in [7]. Various aspects of such methods have been discussed in [1, 2, 3, 4, 5, 6, 15, 16]. Numerical experiments reported in [1, 2, 3, 6, 15, 16] show that the generalized Schwarz algorithms with appropriate parameters can accelerate the convergence dramatically. The aim of this paper is to study the convergence rate of a generalized additive Schwarz algorithm for model problems. A quantitative analysis of the convergence rate for model Dirichlet problems and some numerical results are presented in this paper.

The paper is organized as follows: In Section 2, we introduce a generalized additive Schwarz algorithm. In Sections 3 and 4, we discuss the convergence rate of the generalized additive Schwarz algorithm for one and two dimensional Dirichlet problem respectively. Finally, in Section 5, we give some preliminary numerical results.

2. A Generalized Additive Schwarz Algorithms

Before proceeding, let us introduce a generalized version of additive Schwarz algorithm. We consider the Dirichlet problem for a second order elliptic operator $\mathcal{L}$:

$$\begin{cases} \mathcal{L}u(x) = 0, & x \in \Omega, \\ u(x) = \psi(x), & x \in \partial\Omega, \end{cases} \tag{2.1}$$

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where, Ω is a bounded domain in d-dimensional space ($d = 1, 2, 3$), $\partial\Omega$ is the boundary of Ω, ψ is a given function of $L^2(\Omega)$ and $x = (x_1, \ldots, x_d)$ is the independent variable. To simplify discussion, we consider a case for two subdomians. We also assume that the solution to this problem exists and is unique.

We decompose the solution domain Ω into two overlapping subdomains Ω_1 and Ω_2. That is $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_{12} = \Omega_1 \cap \Omega_2 \neq \emptyset$. Denote by $\Gamma_1 = \partial\Omega_1 \cap \Omega$ and $\Gamma_2 = \partial\Omega_2 \cap \Omega$ the inner boundaries of Ω_1 and Ω_2, respectively.

Denote u_1 and u_2 as the restrictions of the solution u of problem (2.1) on subdomain $\overline{\Omega}_1$ and $\overline{\Omega}_2$, respectively. Then, the following couplings

$$g_1(u_1)|_{\Gamma_1} = g_1(u_2)|_{\Gamma_1}$$

and

$$g_2(u_2)|_{\Gamma_2} = g_2(u_1)|_{\Gamma_2}$$

are true on the inner boundaries Γ_1 and Γ_2, where

$$g_i(v) = \alpha_i v + \beta_i \frac{\partial v}{\partial n_i}, \qquad i = 1, 2. \tag{2.2}$$

Here, for $i = 1, 2$, $\alpha_i \in [0, 1]$, $\beta_i = 1 - \alpha_i$, and n_i is the outer unit normal direction of $\partial\Omega_i$.

With these new couplings we can formulate two coupled subproblems as follows:

$$\begin{cases} \mathcal{L}u_1(x) = 0, & x \in \Omega_1, \\ u_1(x) = \psi(x), & x \in \partial\Omega_1 \cap \partial\Omega, \\ g_1(u_1(x)) = g_1(u_2(x)), & x \in \Gamma_1, \end{cases} \tag{2.3}$$

$$\begin{cases} \mathcal{L}u_2(x) = 0, & x \in \Omega_2, \\ u_2(x) = \psi(x), & x \in \partial\Omega_2 \cap \partial\Omega, \\ g_2(u_2(x)) = g_2(u_1(x)), & x \in \Gamma_2. \end{cases} \tag{2.4}$$

We have the following result (see in [15]).

Theorem 2.1. *If the boundary value problem*

$$\begin{cases} \mathcal{L}w(x) = 0, & x \in \Omega_{12}, \\ w(x) = 0, & x \in \partial\Omega_{12} \backslash (\Gamma_1 \cap \Gamma_2), \\ g_1(w(x)) = 0, & x \in \Gamma_1, \\ g_2(w(x)) = 0, & x \in \Gamma_2 \end{cases} \tag{2.5}$$

has only trivial solution and the solutions u_1, u_2 of (2.3) and (2.4) exist, then

1. *$u_1(x) = u_2(x)$ if $x \in \Omega_{12}$.*
2. *$u(x) = u_1(x)$ if $x \in \Omega_1$ and $u(x) = u_2(x)$ if $x \in \Omega_2$,*

where u is the solution of (2.1).

The original form of additive Schwarz algorithm consists of the following steps.

Let u^0 be an initial function defined on $\bar{\Omega}$ such that $u^0 - \psi$ vanishes on $\partial\Omega$. Set $u_1^0 = u^0|_{\overline{\Omega}_1}$, $u_2^0 = u^0|_{\overline{\Omega}_2}$. For $k > 0$, we define independently the following two sequences respectively:

$$\begin{cases} \mathcal{L}u_1^{k+1}(x) = 0, & x \in \Omega_1, \\ u_1^{k+1}(x) = \psi(x), & x \in \partial\Omega_1 \cap \partial\Omega, \\ u_1^{k+1}(x) = u_2^k(x), & x \in \Gamma_1, \end{cases} \tag{2.6}$$