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ON KARUSH-KUHN-TUCKER POINTS FOR A SMOOTHING METHOD IN SEMI-INFINITE OPTIMIZATION *1)

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Abstract

We study the smoothing method for the solution of generalized semi-infinite optimization problems from (O. Stein, G. Still: Solving semi-infinite optimization problems with interior point techniques, SIAM J. Control Optim., 42(2003), pp. 769–788). It is shown that Karush-Kuhn-Tucker points of the smoothed problems do not necessarily converge to a Karush-Kuhn-Tucker point of the original problem, as could be expected from results in (F. Facchinei, H. Jiang, L. Qi: A smoothing method for mathematical programs with equilibrium constraints, Math. Program., 85(1999), pp. 107–134). Instead, they might merely converge to a Fritz John point. We give, however, different additional assumptions which guarantee convergence to Karush-Kuhn-Tucker points.

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1. Introduction

This article studies a numerical solution method for so-called generalized semi-infinite optimization problems. These problems have the form

$$GSIP: \qquad \text{minimize} \quad f(x) \quad \text{subject to} \quad x \in M$$

with

$$M \;=\; \{\; x \in \mathbb{R}^n |\; g_i(x,y) \le 0 \quad \text{for all } y \in Y(x),\; i \in I \;\}$$

and

$$Y(x) \;=\; \{\; y \in \mathbb{R}^m |\; v_\ell(x,y) \le 0,\; \ell \in L \;\}.$$

All defining functions $f, g_i\,,\; i \in I = \{1,...,p\},\; v_\ell\,,\; \ell \in L = \{1,...,s\}$, are assumed to be real-valued and d times continuously differentiable on their respective domains with $d \ge 2$. The inclusion of equality constraints in the definitions of M and $Y(x)$ as well as of $i-$dependent index sets $Y(x)$ is straightforward and will not be considered here for the ease of presentation.

As opposed to a standard semi-infinite optimization problem *SIP*, the possibly infinite index set $Y(x)$ of inequality constraints is x-dependent in a *GSIP*. For surveys about *standard* semi-infinite optimization we refer to [6, 8, 17, 18], whereas the state of the art in *generalized* semi-infinite optimization is covered in [26, 27, 28] and in the monography [24]. The latter also contains a wide range of applications and the historical background of generalized semi-infinite programming.

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A numerical solution method for a subclass of these problems was presented in [26]. It bases on a smoothing method which is also known from [2] for mathematical programs with complementarity constraints and which is essentially an interior point approach for a degenerate part of the problem. Section 2 explains the main features of this method.

In [26] we have shown that under weak assumptions global solutions of the smoothed problems converge to a global solution of *GSIP*, and that stationary points in the sense of Fritz John converge to a Fritz John point of *GSIP*. From the results in [2] it could be expected that without further assumptions even Karush-Kuhn-Tucker points of the smoothed problems converge to a Karush-Kuhn-Tucker point of *GSIP*.

The aim of the present article is to show that in the setting of *GSIP* this is actually *not* the case. We give, however, different additional assumptions which guarantee the convergence to a Karush-Kuhn-Tucker point. These are the contents of Sections 3 and 4.

2. Preliminaries

This section reviews the main ideas of the smoothing method from [26].

2.1. The Reduction Ansatz for convex lower level problems

The n-parametric so-called lower level problems of *GSIP* are given by

$$Q^i(x): \qquad \underset{y\in\mathbb{R}^m}{\text{maximize}} \quad g_i(x,y) \quad \text{subject to} \quad y \in Y(x)$$

with $i \in I$. Note that the upper level decision variable x is a parameter of the lower problem, and that the upper level index variable y is the decision variable of the lower level. For each parameter value x we can study the optimal value and the optimal points of the optimization problem $Q^i(x)$. More precisely, associated with $Q^i(x)$ are its optimal value function

$$\varphi_i(x) \;=\; \begin{cases} \sup\limits_{y\in Y(x)} g_i(x,y), & \text{if} \quad Y(x) \neq \emptyset \\ -\infty, & \text{else,} \end{cases}$$

and, in case of solvability, its solution set mapping

$$Y^i_\star(x) \;=\; \{y \in Y(x) |\; g_i(x,y) = \varphi_i(x)\}.$$

It is easily seen that M and the set $\{x \in \mathbb{R}^n |\; \varphi_i(x) \leq 0,\; i \in I\}$ coincide.

Assumption 2.1. *For all $x \in \mathbb{R}^n$ the lower level problems $Q^i(x),\; i \in I$, are convex, that is, the functions $-g_i(x,\cdot),\; v_\ell(x,\cdot)$, $\ell \in L$, are convex on $\mathbb{R}^m$.*

Assumption 2.2. *For all $x \in \mathbb{R}^n$ the sets $Y(x)$ are bounded and satisfy the Slater condition, that is, there exists some $y^\star$ such that $v_\ell(x,y^\star) < 0$ for all $\ell \in L$.*

Under Assumptions 2.1 and 2.2 the sets $Y^i_\star(x)$ are nonempty and locally bounded around each $\bar{x} \in \mathbb{R}^n$ ([10]), so that the optimal value functions $\varphi_i(x) \;=\; \max_{y\in Y(x)} g_i(x,y),\; i \in I$, are well-defined and continuous on $\mathbb{R}^n$ ([10]). In particular the feasible set M is closed.

For the derivation of stationarity conditions we concentrate on the nontrivial case of a point $\bar{x}$ from the boundary ∂M of M. Let $I_0(\bar{x}) = \{i \in I |\; \varphi_i(\bar{x}) = 0\}$ denote the set of active indices