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A NONMONOTONE SECOND-ORDER STEPLENGTH METHOD FOR UNCONSTRAINED MINIMIZATION *1)

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Abstract

In this paper, a nonmonotone method based on McCormick's second-order Armijo's step-size rule [7] for unconstrained optimization problems is proposed. Every limit point of the sequence generated by using this procedure is proved to be a stationary point with the second-order optimality conditions. Numerical tests on a set of standard test problems are presented and show that the new algorithm is efficient and robust.

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1. Introduction

Consider the unconstrained optimization problem

$$\min_{x\in R^n} f(x), \tag{1.1}$$

where $f(x)$ is a real-valued twice continuously differentiable function.

There are two classes of basic global approaches to solve problem (1.1): the line search method and the trust region method. Most of these methods naturally require monotone decrease of the objective values to guarantee the global convergence. However, this usually slows the convergence rate of the minimization process, especially in the presence of steep-sided valleys. Recently, several algorithms with nonmonotone techniques have been proposed both in line search methods [5, 6, 11, 16], and trust region methods [3, 4, 10, 15]. Theoretical properties and numerical tests show that the nonmonotone techniques are efficient and competitive [12].

In [7] McCormick modified Armijo's rule and proposed a second-order Armijo's step-size rule, which includes second-order derivative information in the line-search. Using directions of negative curvature, this method can handle the cases where the Hessian matrices are not positive definite, so that the sequence generated by this method converges to a second-order stationary point.

Nonmonotone techniques now are proved to be popular and efficient to deal with optimization problems, especially for ill-conditioned optimization problems. In this paper, we will combine the nonmonotone technique with the second-order Armijo's step-size rule to form a nonmonotone version of the second-order steplength method for unconstrained minimization.

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We first introduce some standard notations used throughout our paper:

1. The notation $\|\cdot\|$ denotes the Euclidean norm on R^n.

2. $g(x) \in R^n$ is the gradient of $f(x)$ evaluated at x, and $H(x) \in R^{n\times n}$ is the Hessian of $f(x)$ at x.

3. If $\{x_k\}$ is a sequence of points generated by an algorithm, we denote $f_k = f(x_k), g_k = g(x_k)$ and $H_k = H(x_k)$.

4. $\lambda_{min}(\cdot)$ stands for the minimal eigenvalue of a matrix.

This paper is organized as follows. In section 2, we describe a nonmonotone algorithm model with the second-order steplength rule and discuss how to determine the descent pair. In section 3 we prove the global convergence which establishes that each limit point of the sequence generated from our algorithm is the second-order stationary point. The numerical results by solving a set of standard test problems are presented in section 4. Finally, in section 5, we give the conclusions.

2. The Nonmonotone Second-order Steplength Method

First of all, we give the definitions of the indefinite point and the descent pair.

Definition 2.1. *A point x is an indefinite point if $H(x)$ has at least one negative eigenvalue. Further, if x is an indefinite point, then d is a direction of negative curvature if $d^T H(x)d < 0$.*

Definition 2.2. *If $s^T g(x) \le 0, d^T g(x) \le 0, d^T H(x)d < 0$, then (s,d) is called a descent pair at the indefinite point x; if x is not an indefinite point and $s^T g(x) < 0, d^T g(x) \le 0$, $d^T H(x)d = 0$, then (s,d) is called a descent pair with zero curvature direction.*

Obviously, when $H(x)$ is positive definite, d must be a zero vector and we only need to consider the descent direction s.

MoCormick's second-order Armijo's step rule is to find the smallest nonnegative integer $i(k)$ from $0, 1, \cdots$, when H_k is indefinite, such that

$$f(y_k(i)) - f(x_k) \le \rho 2^{-i}(s_k^T g_k + \frac{1}{2} d_k^T H_k d_k), \tag{2.1}$$

where

$$y_k(i) = x_k + s_k 2^{-i} + d_k 2^{-i/2}, \tag{2.2}$$

$0 < \rho < 1$ is a preassigned constant and (s_k, d_k) is a descent pair. Then set

$$x_{k+1} = y_k(i(k)).$$

In fact, no matter whether H_k is indefinite or not, we can use the rule (2.1) in every iteration because we can let d_k be a zero vector whenever H_k is positive definite. Clearly, when H_k is positive definite, the second-order step-size rule (2.1) is reduced to the classical Armijo's step rule. In the following, we will assume that the rule (2.1) is used in every iteration.

In order to satisfy (2.1) for a finite integer $i(k)$, it is sufficient that

$$s_k^T g_k < 0$$

whenever $g_k \ne 0$, and

$$d_k^T H_k d_k < 0$$

whenever $g_k = 0$. Such a descent pair (s_k, d_k) does not exist only when x_k is a second-order stationary point. In this case the algorithm will be terminated.

In [7], it is supposed that the second-order step-size rule is used in conjunction with a non-ascent algorithm. In fact, this is not necessary, and it may cause severe loss of efficiency. We can relax the accepting condition on $y_k(i)$. Let

$$f(x_{l(k)}) = \max_{0\le j\le m(k)} f(x_{k-j}), \tag{2.3}$$