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MULTIGRID ALGORITHM FOR THE COUPLING SYSTEM OF NATURAL BOUNDARY ELEMENT METHOD AND FINITE ELEMENT METHOD FOR UNBOUNDED DOMAIN PROBLEMS*[1)]

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Abstract

In this paper, some V-cycle multigrid algorithms are presented for the coupling system arising from the discretization of the Dirichlet exterior problem by coupling the natural boundary element method and finite element method. The convergence of these multigrid algorithms is obtained even with only one smoothing on all levels. The rate of convergence is found uniformly bounded independent of the number of levels and the mesh sizes of all levels, which indicates that these multigrid algorithms are optimal. Some numerical results are also reported.

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Key words: Multigird algorithm, Finite element method, Boundary element method, Coupling, Unbounded domain problem.

1. Introduction

In many fields of scientific and engineering computing, it is necessary to solve boundary values problems of partial differential equations over unbounded domains. The standard techniques such as the finite element method, which is effective for problems in bounded domain, may meet some difficulties for unbounded domain problems and in particular the corresponding computing cost may be very high. So for problems of this kind, it is a good choice to use the method that combines the boundary element method and finite element method. This treatment enables us to combine the advantages of boundary element method for treating domains extended to infinity with those of finite element method in treating the complicated bounded domain problems. Research in this direction is of great importance in both theory and practical computation.

The procedure of this kind of coupling can be briefly described as follows. The unbounded domain is divided into two subregions, i.e., a bounded inner one and an unbounded outer one, by introducing an artificial common boundary. Then, the problem is reduced to an equivalent one in the bounded region. There are many approaches to accomplish this reduction (refer to [6, 7, 8, 10, 11, 12, 14, 15, 17, 19, 20, 24, 26] and references therein). Natural boundary reduction method is one of them.

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Natural boundary reduction method and its coupling with finite element method, which is also known as the exact artificial boundary condition method, were suggested and developed first by Feng in 1980, Yu in 1982 and Han in 1985. And a very similar method, the so-called DtN method, was devised by Keller and Givoli in 1989. In this reduction, the problem over unbounded exterior domain is reduced to an bounded problem with a hyper-singular integral equation on the artificial boundary by using a Green function to get the exact artificial boundary condition with hyper-singular integrals. It is fully compatible with the variational principle over the domain, and the boundary elements are also fully compatible with the domain elements. This coupling is natural and direct. Moreover, the coupled bilinear form preserves automatically the symmetry and coerciveness of the original bilinear form. As a result, the analysis of the discrete problem is simplified, and also the error estimates and the numerical stability are restored (see [11, 23, 24]). In this paper, we follow this approach.

With a discretization scheme, the construction of efficient algorithms for solving the resulting discrete system is of great importance. So, our goal is to construct efficient algorithms for the discrete system obtained from the coupling of natural boundary element method and finite element method.

It is well known that multigrid algorithms are among the most efficient methods for solving discretization equations arising from various finite element approximations of boundary value problem on bounded domain (for multigrid method, refer to [1, 2, 3, 4, 13, 21] and references therein). During the last three decades, there has been intensive research toward multigrid methods. The purpose of this paper is to construct multigrid algorithms for discretization equations arising from the coupling of the natural boundary element method and finite element method for the Dirichlet exterior problem and to investigate their convergence.

In the following sections, some V-cycle multigrid algorithms are constructed. We will investigate the convergence of these multigrid algorithms even with only one smoothing on all levels. The rate of convergence is shown to be uniformly bounded independent of the number of levels and the mesh sizes of all levels, which indicates that the proposed multigrid algorithms are optimal.

The remainder of this paper is organized as follows: In section 2, we present our model problem and introduce the natural boundary reduction method. Multigrid algorithm is described and analyzed in section 3. And some numerical results are reported in section 4.

2. Model Problem and Natural Boundary Reduction

We adopt the standard notations for Sobolev space, with their norms and semi-norms as presented in [5, 9]. Let Ω be a Lipschitz bounded domain in $\mathbb{R}^2$, $\Omega^c = \mathbb{R}^2 \setminus (\Omega \cup \partial\Omega)$, $f \in L^2(\Omega^c)$ be a given compactly supported function. We consider the following model problem

$$\begin{cases} -\triangle u &= f, \quad in \ \ \Omega^c, \\ \quad u &= 0, \quad on \ \ \partial\Omega, \end{cases} \tag{2.1}$$

subject to the asymptotic conditions

$$u(x,y) = \alpha + O(1/r), \qquad\qquad |\nabla u(x,y)| = O(1/r^2),$$

as $r = \sqrt{x^2+y^2} \to \infty$ where α is a constant. Define

$$H^1_\triangle(\Omega^c) = \{v| \ \ \frac{v}{\sqrt{r^2+1}\ln(r^2+2)}\ , \frac{\partial v}{\partial x}\ , \frac{\partial v}{\partial y} \in L^2(\Omega^c)\ ,\ v|_{\partial\Omega} = 0\}$$

and

$$a(w,v) = \int\int_{\Omega^c} \nabla w \cdot \nabla v dxdy\ , \qquad \forall w, v \in H^1_\triangle(\Omega^c)\ .$$