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EXACT AND DISCRETIZED DISSIPATIVITY OF THE PANTOGRAPH EQUATION *1)

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Abstract

The analytic and discretized dissipativity of nonlinear infinite-delay systems of the form $x'(t) = g(x(t), x(qt))(q \in (0,1), t > 0)$ is investigated. A sufficient condition is presented to ensure that the above nonlinear system is dissipative. It is proved the backward Euler method inherits the dissipativity of the underlying system. Numerical examples are given to confirm the theoretical results.

Mathematics subject classification: 65L05.
Key words: Infinite delay, Pantograph equation, Backward Euler method, Dissipativity.

1. Introduction

Let H be a complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the corresponding norm, X a dense continuously imbedded subspace of H. Consider the delay differential equations (DDEs)

$$\begin{cases} x'(t) = g(t, x(t), x(\alpha(t))), \quad t \geq 0, \\ x(t) = \varphi(t), t \in [\inf\limits_{s \geq 0} \alpha(s), 0], \end{cases} \tag{1}$$

where $g : [0, +\infty) \times X \times X \to H, \varphi(t)$ and $\alpha(t)$ are given functions with $\alpha(t) \leq t$ for all $t \geq 0$.

Many dynamical systems are characterized by the property of possessing a bounded absorbing set which all trajectories enter in finite time and thereafter remain inside. In the study of dissipative systems it is often the asymptotic behaviour of the system that is of interest, and so it is highly desirable to have numerical methods that retain the dissipativity of the underlying system.

In 1994, Humphries and Stuart[5, 6] first studied the dissipativity of Runge-Kutta methods for dynamical systems without delay. Later, many results on the dissipativity of numerical methods for dynamical systems without delays were found[7, 8, 20]. For DDEs with constant delay, i.e., $\tau(t) \equiv \tau$, Huang[9, 10] gave a sufficient condition for the dissipativity of the theoretical solution, and investigated the dissipativity of (k, l)-algebraically stable[3] Runge-Kutta methods and $G(c, p, 0)$-algebraically stable[13] one-leg methods. In 2004, Tian[18] studied the dissipativity of DDEs with a bounded variable lag and the numerical dissipativity of θ-method. Moreover, Wen (Wen L.P., Numerical stability analysis for nonlinear Volterra functional differential equations in abstract spaces(in Chinese), Ph.D.Thesis, Xiangtan University, 2005.) discussed the dissipativity of Volterra functional differential equations, and further investigated the disspativity of DDEs with piecewise delays and a class of linear multistep methods.

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An interesting case of (1) is the pantograph equation, corresponding to

$$\alpha(t) = qt, q \in (0, 1),$$

which can be viewed as a representative of infinite time delay. The pantograph equation arises in quite different fields of pure and applied mathematics such as number theory, dynamical systems, probability, mechanics and electrodynamics[2, 11]. In particular, it was used by Ockendon and Tayler[17] to study how the electric current is collected by the pantograph of an electric locomotive, from where it gets its name.

In early work, a constant stepsize was considered for discretization of pantograph equations. As pointed out in Liu[15, 16], however, this kind of stepsize precludes long time integration due to computer memory restrictions. In order to overcome this difficulty, Liu[15] transformed the pantograph equation into a differential equation with a constant delay by a change of variable, suggested by Jackiewicz [12]. Later, Liu[16] and Bellen, Guglielmi and Torelli [1] proposed non-constant stepsize strategies where the stepsizes are geometrically increasing and they investigated the stability of the θ-method.

Recently, many papers have dealt with exact and discretized stability of pantograph equations (see, e.g.,[1, 11, 16]). But up to now, no results of dissipativity have been known for the pantograph equation and its discrete counterpart.

In this paper, we transform the pantograph equation into a non-autonomous DDE with a constant delay by a change of variable, then investigate the dissipativity of the resulting DDE and the backward Euler method. A sufficient condition is presented to ensure that the above system is dissipative. It is shown that the backward Euler method inherits the dissipativity of the underlying system.

2. Dissipativity of DDEs

Consider pantograph equation

$$\begin{cases} x'(t) = g(x(t), x(qt)), & t \geq 0, \\ x(0) = x_0, \end{cases} \tag{2}$$

where q is a constant with $0 < q < 1$, and g satisfies

$$\mathrm{Re}\langle u, g(u,v)\rangle \leq \gamma + \alpha\|u\|^2 + \beta\|v\|^2, \quad u, v \in X, \tag{3}$$

with γ, α and β denoting real constants.

By the change of the independent variable $y(t) = x(e^t)$(see [12, 15]), (2) can be transformed into the constant delay differential equation

$$\begin{cases} y'(t) = f(t, y(t), y(t-\tau)), & t \geq 0, \\ y(t) = \varphi(t), \quad t \leq 0, \end{cases} \tag{4}$$

where $\tau = -\log q$ and

$$f(t, y(t), y(t-\tau)) = e^t g(y(t), y(t-\tau)). \tag{5}$$

It follows from (3) and (5) that

$$\mathrm{Re}\langle u, f(t,u,v)\rangle \leq e^t(\gamma + \alpha\|u\|^2 + \beta\|v\|^2), \quad t \geq 0, u, v \in X. \tag{6}$$

Definition 1. *The evolutionary equation (2) is said to be dissipative in H if there is a bounded set $\mathcal{B} \subset H$ such that for all bounded sets $\Phi \subset H$ there is a time $t_0 = t_0(\Phi)$, such that for all initial values x_0 contained in Φ, the corresponding solution $x(t)$ is contained in $\mathcal{B}$ for all $t \geq t_0$. $\mathcal{B}$ is called an absorbing set in H.*