Journal of Computational Mathematics, Vol.25, No.1, 2007, 89–96.

LIMITED MEMORY BFGS METHOD FOR NONLINEAR MONOTONE EQUATIONS *1)

Weijun Zhou
(*College of Mathematics, Changsha University of Science and Technology, Changsha 410077, China
Email: weijunzhou@126.com*)
Donghui Li
(*College of Mathematics and Econometrics, Hunan University, Changsha 410082, China
Email: dhli@hnu.cn*)

Abstract

In this paper, we propose an algorithm for solving nonlinear monotone equations by combining the limited memory BFGS method (L-BFGS) with a projection method. We show that the method is globally convergent if the equation involves a Lipschitz continuous monotone function. We also present some preliminary numerical results.

Mathematics subject classification: 90C30, 65K05.
Key words: Limited memory BFGS method, Monotone function, Projection method.

1. Introduction

In this paper, we consider the problem of finding a solution of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where $F : R^n \to R^n$ is continuous and monotone. By monotonicity, we mean

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in R^n.$$

Nonlinear monotone equations have strong practical background, which include the subproblems in the generalized proximal algorithms with Bregman distances [10], the first order necessary condition of the unconstrained convex optimization problem and the KKT system of the convex equality constrained convex optimization problem. Some monotone variational inequality problems can also be converted into nonlinear monotone equations by means of fixed point maps or normal maps [22].

Among numerous algorithms for solving systems of smooth equations, the Newton method, quasi-Newton methods, Levenberg-Marquardt method and their variants are particularly useful because of their fast local convergence property [5, 6, 7, 19]. A general way to enlarge the convergence domain of the algorithm is to introduce some line search strategy such that the generated iterates exhibits descent property for some merit function [11]. Solodov and Svaiter [21] presented a Newton-type algorithm with a hybrid projection method for solving systems of monotone equations. The algorithm is globally convergent. The study on the globally convergent quasi-Newton method for solving nonlinear equations is relatively fewer. The major difficulty is the lack of practical line search strategy. Griewank [8] proposed a globally convergent Broyden's method. By using a nonmonotone line search process, Li and Fukushima [13, 14] proposed a Broyden's method for solving nonlinear equations and a Gauss-Newton-based BFGS method for solving symmetric nonlinear equations. Quite recently, Gu, Li, Qi and Zhou [9]

* Received July 13, 2005; final revised November 7, 2005, accepted March 24, 2006.
1) Support by NSF of China grant 10471036 and a 973 project.

introduced a norm descent line search technique and proposed a norm descent BFGS method for solving symmetric equations with global convergence.

A common drawback of the above mentioned quasi-Newton methods is that they need to compute and store an matrix at each iteration. This is computationally costly for large scale problems. To overcome this drawback, Nocedal [18] proposed a limited memory BFGS method (L-BFGS) for unconstrained optimization problems. Numerical results [4, 16] showed that the L-BFGS method is very competitive due to its low storage. This technique has received much attention in recent years, see, e.g., [1, 2, 3, 12, 17, 23] and references therein. However, as far as we know, there seems no related work for solving nonlinear equations. The purpose of this paper is to develop a L-BFGS method for solving nonlinear equations with monotone functions. The method can be regarded as a combination of the L-BFGS method [18] and the projection method [21]. Under some mild assumptions, we prove the global convergence of the method.

In Section 2, we state the steps of the method. In Section 3, we establish the global convergence of the method. In Section 4, we report some preliminary numerical results.

2. Algorithm

In this section, we describe the details of the method. First, we briefly review the L-BFGS method for solving the unconstrained optimization problem:

$$\min f(x), \quad x \in R^n,$$

where $f : R^n \to R$ is continuously differentiable. We denote by $\nabla f(x)$ the gradient of f at x. The steps of the L-BFGS method [16] for solving the unconstrained optimization problem are stated as follows.

Algorithm 1 (L-BFGS algorithm).

Step 1: Given initial point $x_0 \in R^n$, integer m and a symmetric positive definite matrix B_0. Let $k := 0$.

Step 2: Compute d_k by $B_k d_k = -\nabla f(x_k)$, $x_{k+1} = x_k + \alpha_k d_k$, where α_k satisfies some line search.

Step 3: Let $\tilde{m} = \min\{k+1, m\}$. Choose a symmetric and positive definite matrix $B_k^{(0)}$ and a set of increasing integers $L_k = \{j_0, \cdots, j_{\tilde{m}-1}\} \subseteq \{0, \cdots, k\}$. Update $B_k^{(0)}$ $\tilde{m}$ times using the pairs $\{y_{jl}, s_{jl}\}_{l=0}^{\tilde{m}-1}$, i.e., for $l = 0, \cdots, \tilde{m}-1$ compute

$$B_k^{(l+1)} = B_k^{(l)} - \frac{B_k^{(l)} s_{jl} s_{jl}^T B_k^{(l)}}{s_{jl}^T B_k^{(l)} s_{jl}} + \frac{y_{jl} y_{jl}^T}{y_{jl}^T s_{jl}},$$

where $s_k = x_{k+1} - x_k$ and $y_k = \nabla f(x_{k+1}) - \nabla f(x_k)$. Set $B_{k+1} = B_k^{(\tilde{m})}$ and $k := k+1$. Go to Step 2.

There are many possible choice of $B_k^{(0)}$ in Step 3, for example, we can let $B_k^{(0)} = B_0$.

To describe our method, let us recall the projection method [21] for solving the nonlinear monotone equation (1.1). By the monotonicity of F, we have

$$\langle F(z_k), \bar{x} - z_k \rangle \leq 0$$

for any $\bar{x}$ satisfying $F(\bar{x}) = 0$. Suppose we have obtained a direction d_k. By performing some kind of line search procedure along the direction d_k, a point $z_k = x_k + \alpha_k d_k$ can be computed such that

$$\langle F(z_k), x_k - z_k \rangle > 0.$$