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A NEW APPROACH AND ERROR ANALYSIS FOR RECONSTRUCTING THE SCATTERED WAVE BY THE POINT SOURCE METHOD [*1)]

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Abstract

Consider an inverse scattering problem by an obstacle $D \subset \mathcal{R}^2$ with impedance boundary. We investigate the reconstruction of the scattered field u^s from its far field pattern u^∞ using the point source method. First, by applying the boundary integral equation method, we provide a new approach to the point-source method of Potthast by classical potential theory. This extends the range of the point source method from plane waves to scattering of arbitrary waves. Second, by analyzing the behavior of the Hankel function, we obtain an improved strategy for the choice of the regularizing parameter from which an improved convergence rate (compared to the result of [15]) is achieved for the reconstruction of the scattered wave. Third, numerical implementations are given to test the validity and stability of the inversion method for the impedance obstacle.

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1. Introduction

Let $D \subset \mathcal{R}^2$ be a domain with smooth boundary $\partial D \in C^2$ such that the exterior $\mathcal{R}^2 \setminus \overline{D}$ is connected. If we consider D as a 2-D impenetrable obstacle with impedance boundary, then, for a given incident wave $u^i(x)$ such as an incident plane wave $e^{ikx\cdot d}$ with incident direction $d \in \Omega = \{\xi \in \mathcal{R}^2, |\xi| = 1\}$ and wave number $k > 0$, the total wave field

$$u(x) = u^i(x) + u^s(x) \tag{1.1}$$

with the scattered wave field $u^s(x)$ outside D is governed by ([1], Ch.3)

$$\begin{cases} \Delta u + k^2 u = 0, & x \in \mathcal{R}^2 \setminus \overline{D}, \\ \frac{\partial u(x)}{\partial \nu(x)} + ik\sigma(x)u(x) = 0, & x \in \partial D, \\ \frac{\partial u^s(x)}{\partial r} - iku^s(x) = O(\frac{1}{\sqrt{r}}), & r = |x| \to \infty. \end{cases} \tag{1.2}$$

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Here, $\nu(x)$ is the outward normal direction on ∂D, and $0 \le \sigma(x) \in C(\partial D)$ is the boundary impedance for D.

For the scattering problem (1.1), it is well known that the scattered wave field $u^s(x)$ has the asymptotic expression([5, 6])

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left[u^\infty(\hat{x}) + O\left(\frac{1}{\sqrt{|x|}} \right) \right], \quad |x| \to \infty, \quad \hat{x} = \frac{x}{|x|} \in \Omega, \tag{1.3}$$

where $u^\infty(\hat{x})$ is called the far-field pattern of the scattered wave field. Both direct and inverse scattering problems have a long history. The direct problem is to determine the scattered wave as well as its far-field pattern for a known scatterer and incident wave, while the inverse problem is to recover a scatterer D from given information about $u^s(x)$. The typical inverse scattering problem is to determine ∂D from the far-field pattern u^∞ of $u^s(x)$. For the scattering described by (1.1) and (1.2), some related results may be found in ([1, 3, 6, 15]) and the references therein.

The relation between the scattered wave $u^s(x)$ and its far-field pattern $u^\infty(\hat{x})$ is also of great importance, for both direct and inverse scattering problems. On one hand, the far-field pattern $u^\infty(\hat{x})$ for $\hat{x} \in \Omega$ can determine the scattered wave uniquely as stated by the well-known Rellich lemma ([4]), which means that we can determine $u^s(\cdot)$ in $\mathcal{R}^2 \setminus \overline{D}$ from the knowledge of $u^\infty(\cdot)$ given in Ω. On the other hand, the determination of $u^s(x)$ from $u^\infty(\hat{x})$ is ill-posed, that is, the mapping from u^∞ to u^s is unbounded ([15]), which implies that a small perturbation in the far-field data can cause a large error in the scattered wave. Therefore some regularization technique should be introduced, such that we can use the noisy data of u^∞ to determine u^s approximately and stably.

The recovery of u^s from u^∞ has been studied theoretically and numerically for a long time. One method is to express $u^s(x)$ as an infinite series

$$u^s(x) = \sum_{n=0}^{\infty} \sum_{p \in \{\pm 1\}} a_n^p H_n^{(1)}(k|x|) e^{i(pn\varphi)} \tag{1.4}$$

(where $H_n^{(1)}$ denotes the Hankel function of the first kind of order n and φ is the angle between $\hat{x}$ and $(1,0)^T$) with the coefficients determined by $u^\infty(\hat{x}, d)$ ([1], Theorem 3.6, Corollary 3.8). A second method is to establish a relation between $u^s(x)$ and $u^\infty(\hat{x}, d)$ by introducing a density function ([5, 7]). The former method, which expresses the solution $u^s(x)$ explicitly by a recursive relation, is used by engineers widely. However, this method is very sensitive to the noisy far-field pattern data. Also, it has strict geometric limitations, since the recovery of u^s is restricted to the exterior of a circle enclosing the scattering object. The *potential method* of Kirsch and Kress calculates $u^s(x)$ from $u^\infty(\hat{x}, d)$ by solving the integral equation

$$\int_\Gamma \Phi^\infty(\hat{x}, y)\varphi(y)ds(y) = u^\infty(\hat{x}), \quad \hat{x} \in \Omega, \tag{1.5}$$

with some auxillary curve $\Gamma \subset D$, where $\Phi(x,y) = \frac{i}{4} H_0^{(1)}(k|x-y|)$ is the fundamental solution to 2-D Helmholtz equation and $\Phi^\infty(\cdot, y)$ denotes the far field pattern of $\Phi(\cdot, y)$. Please note that in contrast to this notation Φ^∞ is often used for the far field pattern of the scattered field $\Phi^s(\cdot, y)$ for scattering of $\Phi(\cdot, y)$ by some scatterer D. With a solution of (1.5), the scattered field is found by evaluating the potential

$$u^s(x) = \int_\Gamma \Phi(x,y)\varphi(y)ds(y), \quad x \in \mathcal{R}^2 \setminus \overline{D}. \tag{1.6}$$

In a series of papers [11]–[14], a point source method has been proposed to obtain $u^s(x)$ from $u^\infty(\hat{x}, d)$. The main idea of this method as presented in [15] is to approximate the point