Journal of Computational Mathematics, Vol.25, No.2, 2007, 131–144.

HERMITE WENO SCHEMES WITH LAX-WENDROFF TYPE TIME DISCRETIZATIONS FOR HAMILTON-JACOBI EQUATIONS *1)

Jianxian Qiu
(*Department of Mathematics, Nanjing University, Nanjing 210093, China*
Email: jxqiu@nju.edu.cn)

Abstract

In this paper, we use Hermite weighted essentially non-oscillatory (HWENO) schemes with a Lax-Wendroff time discretization procedure, termed HWENO-LW schemes, to solve Hamilton-Jacobi equations. The idea of the reconstruction in the HWENO schemes comes from the original WENO schemes, however both the function and its first derivative values are evolved in time and are used in the reconstruction. One major advantage of HWENO schemes is its compactness in the reconstruction. We explore the possibility in avoiding the nonlinear weights for part of the procedure, hence reducing the cost but still maintaining non-oscillatory properties for problems with strong discontinuous derivative. As a result, comparing with HWENO with Runge-Kutta time discretizations schemes (HWENO-RK) of Qiu and Shu [19] for Hamilton-Jacobi equations, the major advantages of HWENO-LW schemes are their saving of computational cost and their compactness in the reconstruction. Extensive numerical experiments are performed to illustrate the capability of the method.

Mathematics subject classification: 65M06, 65M99, 70H20.
Key words: WENO scheme, Hermite interpolation, Hamilton-Jacobi equation, Lax-Wendroff type time discretization, High order accuracy.

1. Introduction

In this paper, we study an alternative method for time discretization, namely the Lax-Wendroff type time discretization [11], to the popular TVD Runge-Kutta time discretization in [21], for Hermite weighted essentially non-oscillatory (HWENO) schemes [17, 18, 19], termed HWENO-LW schemes, for solving the Hamilton-Jacobi (HJ) equations

$$\begin{cases} \phi_t + H(\nabla_x \phi) = 0, \\ \phi(x,0) = \phi_0(x), \end{cases} \tag{1.1}$$

where $x = (x_1, \cdots, x_d)$ are $d-$spatial variables. The HJ equations appear often in applications, such as in control theory, differential games, geometric optics and image processing. The solutions to (1.1) typically are continuous but with discontinuous derivatives, even if the initial condition $\phi_0(x) \in C^\infty$. It is well known that the HJ equations are closely related to conservation laws, hence successful numerical methods for conservation laws can be adapted for solving the HJ equations. Along this line we mention the early work of Osher and Sethian [13] and Osher and Shu [14] in constructing high order ENO (essentially non-oscillatory) schemes for solving the HJ equations. Central high resolution schemes were developed in [2, 10]. Finite element methods suitable for arbitrary triangulations were developed in [1, 3, 6]. The WENO

* Received May 11, 2006; accepted August 31, 2006.
1) Research partially supported by NNSFC grant 10371118, SRF for ROCS, SEM and Nanjing University Talent Development Foundation.

schemes for solving the HJ equations were constructed in [8] by Jiang and Peng, based on the WENO schemes for solving conservation laws [12, 9, 23], and further Zhang and Shu [24] developed high order WENO schemes on unstructured meshes for solving two-dimensional HJ equations. Finally, most relevant to our work, we mention the HWENO schemes for solving the HJ equations by Qiu and Shu [19], based on the HWENO schemes for solving conservation laws [17, 18].

WENO or HWENO is a spatial discretization procedure, namely, it is a procedure to approximate the spatial derivative terms in (1.1). The time derivative term there must also be discretized. There are mainly two different approaches to approximate the time derivative. The first approach is to use an ODE solver, such as a Runge-Kutta or a multi-step method, to solve the method of lines ODE obtained after spatial discretization. The second approach is a Lax-Wendroff type time discretization procedure, which is also called the Taylor type referring to a Taylor expansion in time or the Cauchy-Kowalewski type referring to the similar Cauchy-Kowalewski procedure in PDE. This approach is based on the idea of the classical Lax-Wendroff scheme [11], and it relies on converting all the time derivatives in a temporal Taylor expansion into spatial derivatives by repeatedly using the PDE and its differentiated versions. The spatial derivatives are then discretized by, e.g. the HWENO approximations.

The Lax-Wendroff type time discretization, usually produces the same high order accuracy with a smaller effective stencil than that of the first approach, and it uses more extensively the original PDE. The original finite volume ENO schemes in [5] used this approach for the time discretization. More recently, a Lax-Wendroff type time discretization procedure for high order finite difference WENO schemes was developed by Qiu and Shu [15]. This approach was also used by Titarev and Toro [22] and Schwartzkopff, et al. [20], termed ADER (arbitrary high order schemes utilizing higher order derivatives), to construct a class of high order schemes for conservation laws in finite volume version. The Lax-Wendroff type time discretization was also used in the discontinuous Galerkin method [4, 16].

In this paper, based on the WENO-LW methodology for conservation laws in [15] and HWENO schemes for HJ equation in [19], we develop HWENO-LW schemes to solve the HJ equations. Comparing with the HWENO-RK schemes of Qiu and Shu [19], the major advantages of HWENO-LW schemes are their saving of computational cost and their compactness in the reconstruction.

The organization of this paper is as follows. In Section 2, we describe in detail the construction and implementation of the HWENO-LW schemes, for one and two-dimensional HJ equations (1.1). In Section 3 we provide extensive numerical examples to demonstrate the behavior of the schemes and to perform a comparison with the HWENO-RK schemes for HJ equations by Qiu and Shu [19]. Concluding remarks are given in Section 4.

2. The Construction of HWENO-LW Schemes for the Hamilton-Jacobi Equations

In this section we will present the details of the construction of HWENO-LW schemes for both one and two-dimensional Hamilton-Jacobi equations.

2.1. One-dimensional Case

We first consider the one dimensional HJ equation (1.1). For simplicity, we assume that the grid points $\{x_{i+1/2}\}$ are uniformly distributed with the cell size $x_{i+1/2} - x_{i-1/2} = \Delta x$ and cell