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THE MAGNETIC POTENTIAL FOR THE ELLIPSOIDAL MEG PROBLEM *1)

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Abstract

In magnetoencephalography (MEG) a primary current is activated within a bounded conductive medium, *i.e.*, the head. The primary current excites an induction current and the total (primary plus induction) current generates a magnetic field which, outside the conductor, is irrotational and solenoidal. Consequently, the exterior magnetic field can be expressed as the gradient of a harmonic function, known as the magnetic potential. We show that for the case of a triaxial ellipsoidal conductor this potential is obtained by using integration along a specific path which is dictated by the geometrical characteristics of the ellipsoidal system as well as by utilizing special properties of ellipsoidal harmonics. The vector potential representation of the magnetic field is also obtained.

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1. Introduction

As Plonsey and Heppner [7] have demonstrated, in studying Bioelectromagnetic problems, the values of the physical parameters of the human body justify the replacement of Maxwell's equations with the equations of quasi-static theory of electromagnetism. This means that the time derivative terms of the magnetic induction and of the electric displacement fields in Maxwell's equations can be omitted. That renders the rotation of the magnetic field proportional to the current. Hence, in regions free of current the magnetic field becomes irrotational and since, due to the lack of magnetic monopoles, it is also solenoidal, it can be represented by the gradient of a harmonic function. This function was first obtained by Bronzan [1] via path integration in appropriate regions that avoid the support of the current. But the actual meaning of the scalar magnetic potential in MEG was demonstrated by Sarvas [8] in a celebrated paper where he showed that for a spherical conductor the exterior magnetic potential can be obtained from the radial component of the primary current alone. In particular, he obtained in closed form the potential and therefore the magnetic field as well, for the case of a homogeneous spherical conductor with a dipole source anywhere in its interior. His solution coincides with the one Bronzan gave for the general case. It is of interest to see though that this property of recovering the exterior magnetic field from the radial component of the primary current is not shared by any other geometry besides the spherical one. In other words, for non spherical conductors the geometry of the conductor influences directly the exterior magnetic field. The ellipsoidal geometry has the advantage of being a genuine three dimensional shape that can be well adjusted in any convex body, and in particular to the brain which anatomicaly is considered to be an ellipsoid with average semiaxes 6, 6.5 and 9 centimeters. On the other hand, it is exactly this freedom of adaptation to any convex body that makes the mathematics much more elaborate than the spherical (1-D) or even the spheroidal (2-D) geometries.

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Obviously, any attempt to calculate the magnetic potential for an ellipsoidal conductor has to incorporate the contribution from the surface distribution of dipoles as given by the Geselowitz formula [5]. Therefore, the direct electroencephalography (EEG) problem has to be solved as well, in order to determine the dipole density function on the boundary of the ellipsoidal conductor. The crucial part of the present work is to determine the path of integration that will allow the calculation of the line integral which provides the magnetic potential. We show that such a path is given by the non planar curve which is defined by the intersection of the one-sheet hyperboloid and the two-sheet hyperboloid that correspond to the "angular" ellipsoidal coordinates of the point where the potential is evaluated. It seams that this choice of integration path is the unique choice which allows for the integration of the ellipsoidal fields. It is the ellipsoidal analogue of the radial direction for the case of a sphere. Following this approach we were able to obtain the exterior magnetic field as a series solution in terms of multipole fields. The leading term of this series, which is the quadrupolic term, was obtained analytically by the author and Kariotou in [2].

We mention here that as far as the inverse MEG problem is concerned, it was shown by Fokas, Kurylev and Marinakis [4] for the sphere and by the author, Fokas and Kariotou [3] for any star-shape conductor, that from the three scalar functions needed to identify the current only one can be recovered, and this is true even when a complete knowledge of the magnetic potential outside the head is provided.

Section 2 states the direct problem of magnetoencephalography for a single dipole in ellipsoidal geometry and provides the solution to the corresponding problem of electroencephalography which concerns the electric potential. Section 3 elaborates a compact expresssion for the multipole expansion of the exterior magnetic field in dyadic form. The vector potential for the magnetic field is discussed in Section 4 while the corresponding scalar magnetic potential is obtained in Section 5.

2. The Ellipsoidal MEG Problem

Consider the ellipsoid

$$\frac{x_1^2}{\alpha_1^2}+\frac{x_2^2}{\alpha_2^2}+\frac{x_3^2}{\alpha_3^2}=1, \tag{2.1}$$

where

$$0<\alpha_3<\alpha_2<\alpha_1<+\infty \tag{2.2}$$

are the three semiaxes and

$$\left.\begin{array}{l} h_1=\sqrt{\alpha_2^2-\alpha_3^2} \\ h_2=\sqrt{\alpha_1^2-\alpha_3^2} \\ h_3=\sqrt{\alpha_1^2-\alpha_2^2} \end{array}\right\} \tag{2.3}$$

are the three semifocal distances. Since

$$h_1^2-h_2^2+h_3^2=0, \tag{2.4}$$

only two out of the three semifocal distances are independent.

Introduce the ellipsoidal coordinates [6] (ρ,μ,ν) via

$$\left.\begin{array}{l} x_1=\frac{1}{h_2h_3}\rho\mu\nu \\ x_2=\frac{1}{h_1h_3}\sqrt{\rho^2-h_3^2}\sqrt{\mu^2-h_3^2}\sqrt{h_3^2-\nu^2} \\ x_3=\frac{1}{h_1h_2}\sqrt{\rho^2-h_2^2}\sqrt{h_2^2-\mu^2}\sqrt{h_2^2-\nu^2} \end{array}\right\} \tag{2.5}$$

where

$$0<\nu^2<h_3^2<\mu^2<h_2^2<\rho^2<+\infty. \tag{2.6}$$