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ANISOTROPIC POLARIZATION TENSORS FOR ELLIPSES AND ELLIPSOIDS [*1)]

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Abstract

In this paper we present a systematic way of computing the polarization tensors, anisotropic as well as isotropic, based on the boundary integral method. We then use this method to compute the anisotropic polarization tensor for ellipses and ellipsoids. The computation reveals the pair of anisotropy and ellipses which produce the same polarization tensors.

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1. Introduction

Consider a field ∇H in $\mathbb{R}^d$, $d = 2, 3$, where H is a harmonic function in $\mathbb{R}^d$. The most important such a field to consider is given by $H(x) = x_j$, $j = 1, \ldots, d$. The field is disturbed by the presence of an inclusion B which is a bounded Lipschitz domain in $\mathbb{R}^d$. Let ∇u be the perturbed field. Then the perturbation admits the multipole expansion

$$(u - H)(x) = \sum_{|\alpha|,|\beta|=1}^{\infty} \frac{(-1)^{|\alpha|}}{\alpha!\beta!} \partial_x^\alpha \Gamma(x) M_{\alpha\beta} \partial^\beta H(0), \quad |x| \to \infty, \tag{1.1}$$

where Γ is the fundamental solution for the Laplacian, and α, β are multi-indices. See [3]. The quantities $M_{\alpha\beta}$, which describe the perturbation of the field completely, are called the generalized polarization tensors (PT). In particular, when $|\alpha| = |\beta| = 1$, then $M = (M_{\alpha\beta})$ is called the first order polarization tensor.

The notion of the polarization tensor can be extended to include the case when the conductivity of the inclusion and that of the matrix (background) are anisotropic. Suppose that the conductivity of the background $\mathbb{R}^d \setminus \overline{B}$ is $\widetilde{\gamma}$, while that of the inclusion B is γ, where $\widetilde{\gamma}$ and γ are positive definite symmetric $d \times d$ constant matrices. After an obvious change of variables we may assume that $\widetilde{\gamma} = I$, the $d \times d$ identity matrix. The matrix γ represents an (anisotropic) material property of the inclusion B. Thus the conductivity profile here is given by $\gamma_B = \chi(\mathbb{R}^d \setminus \overline{B})I + \chi(B)\gamma$, where $\chi(B)$ denotes the characteristic function of B. Here and throughout this paper we assume that $\gamma - I$ is either positive or negative definite. Even in this case the perturbation $(u - H)(x)$ has multipole expansions (1.1) at ∞. We denote the anisotropic polarization tensors (APT) associated with this conductivity distribution γ_B by $M_{\alpha\beta} = M_{\alpha\beta}(I, \gamma; B)$.

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The purpose of this paper is to present a simple method to compute PTs anisotropic as well as isotropic using the layer potential techniques. The method of this paper reduces the computation of PTs to the computation of the boundary integral operator $\mathcal{K}_B$ on ∂B which is defined by (2.6). The method also provides us with a systematic way to compute the PTs numerically. We then proceed to compute anisotropic PTs associated with ellipses and ellipsoids. It has been known that the first order PTs can be realized in terms of ellipses in a unique way if the conductivities of the inclusion and the matrix are isotropic [12, 6]. However, the material property of the inclusion, namely, anisotropic conductivity, cannot be extracted by means of the first order PT. The computations of this paper completely characterize the pairs of anisotropy and an ellipse which yield the same PTs. It should be noted that the first order PTs for the ellipses and ellipsoids are known (see [26]), and the higher order PTs for disks, ellipses, and balls were computed in [24]. The method of this paper has been applied to the computation of the elatstic moment tensor [7].

The notion of the (first order) PT appears in various contexts such as the inverse problems to detect unknown small inclusions, the theory of composite materials, and low frequency asymptotics to name a few. It was Friedman and Vogelius who first used the first order PT for the inverse problem of identifying the location of small inclusions [18]. Since then, various non-iterative direct methods to identify small inclusions using the PT have been proposed and tested numerically [14, 10, 12, 11, 6, 20, 3]. The PT also naturally appears in the asymptotic expansions of effective properties of dilute composite materials [5, 8, 9, 15, 19, 26] and low frequency asymptotics [16, 23]. For a derivation of higher order terms of the asymptotic expansion and extensions to anisotropic and elastic composites, and for extensive references, we refer to a recent book [4]. The notion of PT has been generalized to include all the higher order terms and various important properties of the generalized PT have been obtained [1, 2, 3]. Among them are symmetry, positive-definiteness, the Hashin-Strikman bounds, and the fact that the inclusion is completely determined by all the generalized PT even if the conductivity matrix γ is anisotropic. It should be mentioned that the Hashin-Strikman bounds for the first order PT, which is optimal, was obtained by Lipton [25] and Capdeboscq-Vogelius [13]. The notion of the PT was also used in the study of potential flows by Pólya, Szegö, and Schiffer [27, 28]

This paper is organized as follows. In section 2, we derive a general way to compute PTs using boundary integrals. In section 3, we derive formula for PTs on ellipses. Section 4 is for ellipsoids.

2. Layer Potential Method for Computation of PT

A fundamental solution to the Laplacian is given by

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2, \\ -\dfrac{1}{4\pi} \dfrac{1}{|x|}, & d = 3. \end{cases} \tag{2.1}$$

With this fundamental solution, the single and double layer potentials are defined to be

$$\mathcal{S}_B \phi(x) := \int_{\partial B} \Gamma(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d, \tag{2.2}$$

$$\mathcal{D}_B \phi(x) := \int_{\partial B} \frac{\partial}{\partial \nu_y} \Gamma(x-y)\phi(y)\, d\sigma(y), \quad x \in \mathbb{R}^d \setminus \partial B, \tag{2.3}$$