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SUPERCONVERGENCE OF DG METHOD FOR ONE-DIMENSIONAL SINGULARLY PERTURBED PROBLEMS [*1)]

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Abstract

The convergence and superconvergence properties of the discontinuous Galerkin (DG) method for a singularly perturbed model problem in one-dimensional setting are studied. By applying the DG method with appropriately chosen numerical traces, the existence and uniqueness of the DG solution, the optimal order L_2 error bounds, and $2p+1$-order superconvergence of the numerical traces are established. The numerical results indicate that the DG method does not produce any oscillation even under the uniform mesh. Numerical experiments demonstrate that, under the uniform mesh, it seems impossible to obtain the uniform superconvergence of the numerical traces. Nevertheless, thanks to the implementation of the so-called Shishkin-type mesh, the uniform $2p+1-$order superconvergence is observed numerically.

Mathematics subject classification: 65N30.
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1. Introduction

In scientific and engineering computation, we often encounter differential equations with small parameters and these equations are "singularly perturbed". One of the difficulties in numerically computing the solution of singularly perturbed problems lays in the so-called boundary layer behavior, i.e., the solution varies very rapidly in a very thin layer near the boundary. Traditional methods, such as finite element and finite difference methods, do not work well for these problems as they often produce oscillatory solutions which are inaccurate when the diffusion parameter is small. Numerical simulations of these equations raise very challenging problems for scientists and engineers. There is a rich literature in this direction. The reader is referred to books [14, 15, 18] and survey articles [17, 22] for details. Currently, this is still a very active field, see, e.g., [6, 10, 13, 19]. We have also noticed some publications in this journal on the topic, see, e.g., [12], among others.

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Since the introduction of the first DG method for hyperbolic equations [16], there has been an active development of DG methods for hyperbolic and elliptic equations in parallel, e.g., [1, 7, 8], some recent development [2, 24], and references therein. The model problem in this article is a singularly perturbed convection-diffusion equation in the one-dimensional setting. When the small parameter approaches 0, the problem changes from an elliptic equation to a hyperbolic equation. The superconvergence property of the traditional finite element method under the Shishkin mesh [20] for this model problem was discussed in [26], and the p-version finite element method for this model problem was studied in [21, 25] among others. Inspired by the great success of the DG method in solving hyperbolic equations [7], we adopt it to solve the singularly perturbed convection-diffusion equations.

In this work, we first define the DG scheme by choosing the numerical trace which is very delicate as it can affect the stability and accuracy. Next we verify the existence and uniqueness of the approximate solution. We then focus on the proof of $2p+1$-order superconvergence of the numerical traces at the nodes. For arbitrary ϵ, the uniform convergence is expected. The so-called "uniform convergence" means that the convergence rate is uniformly valid in terms of the singular perturbation parameter ϵ. The numerical results in Section 3 indicate that, under the uniform mesh, it seems impossible to have the uniform $2p+1$-order superconvergence. Nevertheless, an attractive feature is that the DG method does not produce any oscillation outside boundary region even under the uniform mesh. In other words, the DG method is more "local" than the traditional finite element method. On the other hand, when the Shishkin-type meshes are implemented with some appropriately chosen τ, the length of the boundary layer in numerical computation, the uniform superconvergent results are observed in our numerical experiments in Section 3. The theoretical analysis of this exciting phenomenon is an ongoing work. Further, the approach in this work can be generalized to the two-dimensional setting.

During the process of this study, we noticed a parallel work [3], which addressed the same superconvergence issue for the model problem. However, two approaches are completely different and our proof is much simpler. As for general finite element superconvergence theory, we refer readers to following books [4, 5, 9, 11, 23, 27] and references therein.

2. DG Method

Consider the following one-dimensional convection-diffusion problem,

$$\begin{cases} -\epsilon u'' + bu' = f \quad \text{in} \quad (0,1), \\ u(0) = u_0, \quad u'(1) = u_1', \end{cases} \tag{2.1}$$

where $b > 0$ and ϵ is a small positive parameter. The choice of $b > 0$ guarantees that the location of the possible boundary layer is at the outflow boundary $x = 1$.

By setting $q = u'$, (2.1) can be rewritten as

$$\begin{cases} -\epsilon q' + bq = f \quad \text{in} \quad (0,1), \\ q - u' = 0, \\ u(0) = u_0, \quad q(1) = u_1'. \end{cases} \tag{2.2}$$

Denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$ for $j = 1, \cdots, N$ with $x_{\frac{1}{2}} = 0, x_{N+\frac{1}{2}} = 1$. The center of the cell I_j is $x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2$ and $h_j = |I_j|$. Set $h = \max_{1\le j\le N} h_j$ and $\Omega_h = \bigcup_{j=1}^{N} I_j$. We denote by $u^+_{j+\frac{1}{2}}$ and $u^-_{j+\frac{1}{2}}$ the values of u at $x_{j+\frac{1}{2}}$, from the right cell and the left cell of $x_{j+\frac{1}{2}}$, respectively. Denote the jump at $x_{j+\frac{1}{2}}$ by $[u]_{j+\frac{1}{2}} = u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}$.