Journal of Computational Mathematics, Vol.25, No.2, 2007, 201–210.

RADIAL BASIS FUNCTION INTERPOLATION IN SOBOLEV SPACES AND ITS APPLICATIONS*

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Abstract

In this paper we study the method of interpolation by radial basis functions and give some error estimates in Sobolev space $H^k(\Omega)$ $(k \geq 1)$. With a special kind of radial basis function, we construct a basis in $H^k(\Omega)$ and derive a meshless method for solving elliptic partial differential equations. We also propose a method for computing the global data density.

Mathematics subject classification: 41A05, 41A25, 41A30, 41A63.
Key words: Sobolev space, Radial basis function, Global data density, Meshless method.

1. Introduction

For a smooth function u, known only at a set $X = \{x^{(1)}, \cdots, x^{(N)}\}$ consisting of pairwise distinct points in $\mathbb{R}^d$, the corresponding radial basis function approach is the interpolant

$$s_u(x) = \sum_{i=1}^{N} a_i \psi(x - x^{(i)}) + \sum_{l=1}^{Q} b_l p_l(x), \tag{1.1}$$

whose coefficients a_i and b_l are determined by the following linear system

$$\sum_{i=1}^{N} a_i \psi(x^{(j)} - x^{(i)}) + \sum_{l=1}^{Q} b_l p_l(x^{(j)}) = u(x^{(j)}), \qquad j = 1, \cdots, N, \tag{1.2}$$

$$\sum_{j=1}^{N} a_j p_i(x^{(j)}) = 0, \qquad i = 1, \cdots, Q, \tag{1.3}$$

where $Q = C^d_{q+d-1}$, $\psi(x) = \phi(|x|)$, $|.|$ is the Euclidean norm, and $p_1, \cdots, p_Q$ is a basis of $\mathbb{P}_q$, the space of polynomials defined on $\mathbb{R}^d$ with total order $< q$ $(q \geq 0)$. Especially, when $q = 0$, the interpolant (1.1) reads as

$$s_u(x) = \sum_{i=1}^{N} a_i \psi(x - x^{(i)}) \tag{1.4}$$

and the coefficients a_i are determined by

$$\sum_{i=1}^{N} a_i \psi(x^{(j)} - x^{(i)}) = u(x^{(j)}), \qquad j = 1, \cdots, N. \tag{1.5}$$

* Received October 9, 2005; accepted December 6, 2005.

Definition 1.1. ([2]) *A function $F : \mathbb{R}^d \to R$ is said to be conditionally positive definite (resp. strictly conditionally positive definite) of order q $(q \geq 0)$, if for all finite subsets X in $\mathbb{R}^d$, the quadratic form*

$$\sum_{i=1}^{N}\sum_{j=1}^{N} c_i c_j F(x^{(i)} - x^{(j)}) \tag{1.6}$$

is nonnegative (resp. positive) for all vectors (resp. nonzero vectors) $c \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} c_i p(x^{(i)}) = 0$ for any given $p \in \mathbb{P}_q$. If $q = 0$, F is called positive definite (resp. strictly positive definite).

The nonsingularity of system (1.2)-(1.3) for a wide choice of functions ψ and polynomials of order q is assured by

Theorem 1.1. *Let $N \geq Q$. Assume that $\psi(x) = \phi(|x|)$ is strictly conditionally positive definite of order q $(q \geq 0)$. Assume furthermore that there exists a subset $X' \subset X$ containing Q points such that*

$$p|_{X'} = 0 \quad \text{for } p \in \mathbb{P}_q \qquad \text{implies} \quad p \equiv 0. \tag{1.7}$$

Then the interpolation system (1.2)-(1.3) *is always uniquely solvable.*

The proof of this theorem can be found in the appendix of this paper, and readers may refer to Madych [5] and Micchelli [6] for more details about conditionally positive definite functions. We give in Table 1.1 some frequently used radial basis functions, where $\lceil \beta/2 \rceil$ denotes the smallest integer greater than or equal to $\beta/2$, and $\lfloor d/2 \rfloor$ denotes the biggest integer less than or equal to $d/2$.

Table 1.1: Radial basis function

Name	$\psi(x) = \phi(r),\ r = \|x\|$	$\widehat{\psi}(\xi)$	q
Gaussians	$e^{-\beta r^2}, \quad \beta > 0$	$C(d,\beta)e^{-\|\xi\|^2/4\beta}$	0
Thin plate spline	$(-1)^{1+\beta/2} r^\beta \ln r, \quad \beta \in 2\mathbb{N}$ $(-1)^{\lceil \beta/2 \rceil} r^\beta,\ \beta \in \mathbb{R}_{>0} \setminus 2\mathbb{N}$	$C(d,\beta)\|\xi\|^{-d-\beta}$	$1 + \beta/2$ $\lceil \beta/2 \rceil$
Sobolev spline	$K_{\beta-d/2}(r) r^{\beta-d/2},\ \beta > d/2$ K MacDonald's function	$C(d,\beta)(1 + \|\xi\|^2)^{-\beta}$	0
compactly supported functions, C^{2l}	$(1-r)_+^\beta p(r)$ $\partial p = l,\ \beta = \lfloor d/2 \rfloor + 2l + 1$	$(1 + \|\xi\|)^{-d-2l-1}$	0

There is a rapidly growing on list of literatures related to radial basis functions, and the accuracy of interpolation by radial basis functions often comes out very satisfactory. Yet, this satisfaction is based on the presumption that the approximand is reasonably smooth. At least, such interpolations require the pointwise value of the approximand. However, taking functions in H^1 for example, this pointwise value may not be well defined for a wide variety of functions. Although an interpolation approach by radial basis functions in Sobolev spaces is provided in [9, 10, 12], yet, their assumption $k > \frac{d}{2}$ restricts the approximand to be continuous in essential. The approximation of nonsmooth functions by using continuous piecewise polynomials is studied in [7], but regular grid data is required. In this paper, we propose an interpolation of scattered data by radial basis functions in Sobolev space $H^k(\Omega), k \geq 1$, and by means of such interpolation with a special kind of radial basis function, we derive a meshless method for solving elliptic partial differential equations.