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SENSITIVITY ANALYSIS WITH RESPECT TO THE ELECTRICAL CONDUCTIVITY [*1)]

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Abstract

In this paper, we consider conductivity inclusions inside a homogeneous background conductor. We provide a complete asymptotic expansion of the solution of such problems in terms of small variations in the electrical conductivity of the inclusion. Our method is based on a boundary integral perturbation theory. Our results are valid for both high and low contrast inclusions.

Mathematics subject classification: 35B30.
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1. Introduction

An interesting problem arising in the study of photonic band gap structures concerns the calculation of electrostatic properties of systems made by high contrast materials. By high contrast, we mean that the electrical conductivity ratio is high. When the material contrast is high, standard numerical procedures can become ill-conditioned. We refer to Tausch, White, and Wang [10, 11] and Greengard and Lee [6] for effective algorithms for this class of problems. The Tausch-White-Wang approach is based on a perturbation theory while the method of Greengard and Lee is a modification of the classical integral equation.

In this paper, we derive a complete asymptotic expansion of the solution of the conductivity problem due to small variations in the conductivity ratio by a boundary integral perturbation method. We provide error estimates for the approximation. Our results are valid for inclusions with extreme conductivities (zero or infinite conductivity). In particular, our method may be viewed as a different approach which can potentially simplify calculations for problems involving highly conducting inclusions.

Consider a homogeneous conducting object occupying a bounded domain $\Omega \subset \mathbb{R}^2$, with a connected Lipschitz boundary $\partial\Omega$. We assume, for the sake of simplicity, that its conductivity is

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equal to 1. Let D with Lipschitz boundary be a conductivity inclusion inside Ω of conductivity equal to some positive constant $k \neq 1$. Let u_k be the solution of

$$\begin{cases} \nabla \cdot (1 + (k-1)\chi_D)\nabla u_k = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \nu}\Big|_{\partial\Omega} = g \in L^2_0(\partial\Omega), \quad \displaystyle\int_{\partial\Omega} u_k = 0, \end{cases} \tag{1.1}$$

where χ_D is the indicator function of D. We allow k to be 0 or $+\infty$. If $k = 0$, the inclusion D is insulated, and the equation in (1.1) is replaced with

$$\begin{cases} \Delta u_0 = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \dfrac{\partial u_0}{\partial \nu}\Big|_{\partial D} = 0, \quad \dfrac{\partial u_0}{\partial \nu}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega} u_0 = 0, \end{cases}$$

and if $k = +\infty$, then D is a perfect conductor and the equation in (1.1) is replaced with

$$\begin{cases} \Delta u_\infty = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ \nabla u_\infty = 0 \quad \text{in } D, \\ \dfrac{\partial u_\infty}{\partial \nu}\Big|_{\partial\Omega} = g, \quad \displaystyle\int_{\partial\Omega} u_\infty = 0. \end{cases} \tag{1.2}$$

It was proved in [4, 7] that u_k converges in $W^{1,2}(\Omega \setminus \overline{D})$ to u_0 or u_∞ as $k \to 0$ or $k \to +\infty$. Here the space $W^{1,2}(\Omega \setminus \overline{D})$ is the set of functions $f \in L^2(\Omega \setminus \overline{D})$ such that $\nabla f \in L^2(\Omega \setminus \overline{D})$. The main result of this paper is a rigorous derivation, based on layer potential techniques, of a complete asymptotic expansion of $u_k|_{\partial\Omega}$ as $k \to +\infty$ or 0. In fact we will derive an asymptotic formula of $u_k|_{\partial\Omega}$ when $k \to k_0$.

This paper is organized as follows. In the next section we give an explicit asymptotic formula of u_k as $k \to +\infty$ or 0 when Ω is a disk and D is a concentric disk. In Section 3, we derive a complete asymptotic formula for $u_k - u_{k_0}$ on $\partial\Omega$ when $k \to k_0$. The formula is valid even when $k_0 = 0$ or $+\infty$.

2. Explicit Formula

In this section, Ω is assumed to be the unit disk centered at the origin, and D to be the concentric disk centered at the origin with radius α. Set

$$g(1, \theta) = \sum_{n \in \mathbb{Z}\setminus\{0\}} g_n e^{in\theta}.$$

Write

$$u_k = \begin{cases} a_0 + b_0 \ln(r) + \displaystyle\sum_{n \in \mathbb{Z}\setminus\{0\}} (a_n r^{|n|} + b_n r^{-|n|}) e^{in\theta} \quad \text{in } \Omega \setminus \overline{D}, \\ \displaystyle\sum_{n \in \mathbb{Z}} \frac{c_n}{\alpha^{|n|}} r^{|n|} e^{in\theta} \quad \text{in } D, \end{cases}$$

where the Fourier coefficients a_n, b_n and c_n are to be found.

Since $g \in L^2_0(\partial\Omega)$ and $\int_{\partial\Omega} u_k = 0$, we have that $a_0 = b_0 = 0$. Using the continuity of u_k across the interface ∂D, we get $c_0 = 0$. Then, for $n \in \mathbb{Z} \setminus \{0\}$, we have

$$\begin{cases} |n| a_n - |n| b_n = g_n, \\ a_n \alpha^{|n|} + b_n \alpha^{-|n|} - c_n = 0, \\ a_n \alpha^{|n|} - b_n \alpha^{-|n|} - k c_n = 0, \end{cases}$$