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NON SPURIOUS SPECTRAL-LIKE ELEMENT METHODS FOR MAXWELL'S EQUATIONS*

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Abstract

In this paper, we give the state of the art for the so called "mixed spectral elements" for Maxwell's equations. Several families of elements, such as edge elements and discontinuous Galerkin methods (DGM) are presented and discussed. In particular, we show the need of introducing some numerical dissipation terms to avoid spurious modes in these methods. Such terms are classical for DGM but their use for edge element methods is a novel approach described in this paper. Finally, numerical experiments show the fast and low-cost character of these elements.

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Key words: Maxwell's equations, Discontinuous Galerkin methods, Edge elements, Mass lumping.

1. Introduction

For a long time, Maxwell's equations were mainly solved in the time-harmonic domain. The evolution of radar techniques showed the limit of this formulation which can only treat monochromatic sources. Both engineers and researchers were then motivated to use equations in the time domain which can take into account large frequency sources in one resolution. The first and most popular approximation of Maxwell's equations in the time domain was provided by the Yee's scheme [29], commonly called FDTD (Finite Difference in the Time Domain) by engineers, which is basically a centered second order finite difference approximation of Maxwell's equation.

Although easy to implement, FDTD has some difficulties to treat complex geometries. In fact, the staircase approximation of curved boundary can produce spurious reflections which can substantially pollute the solution. On the other hand, finite element methods (FETD) have the major drawback of producing a n-diagonal (n can grow up to several tens in 3D) mass matrix which must be inverted at each time-step, which is a serious handicap for FETD versus FDTD whose mass matrix is the identity matrix. This mass matrix does not present any difficulty to time harmonic problems, for which even the stiffness matrix must be inverted. For this reason, industry was reluctant to use FETD for a long time and FDTD remains the reference for Maxwell's equations in the time domain for 40 years!

The mass lumping technique is an efficient alternative to mass matrix inversion. However, this technique was well known for lower order continuous (or H^1) elements but not obvious for higher-order approximations. A first step towards a general mass lumping technique was made by Hennart et al. [19,20] and independently by Young [30] which proposed to use Gauss-Lobatto quadrature formulas to get mass lumping for continuous quadrilateral or hexahedral

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finite elements. Besides mass lumping, these formulas ensure to keep the order induced by the finite element approximation [3]. First introduced for ODE or parabolic problems, this technique was extended to the wave equation by Cohen et al. [11] and renamed spectral element methods [22]. The non trivial extension of this technique to triangular and tetrahedral elements was later realized by Cohen et al. [10] for triangles up to third order and Mulder et al. [23] for higher-order triangles and tetrahedra.

The problem of the mass matrix inversion was solved for the wave equation and remained a challenging problem for Maxwell's equations. A first try was done by Haugazeau et al. [18] but this approach remains restricted to first-order approximation. A second and natural step was to extend spectral element techniques to edge (or $H(curl)$) elements. This was done by Cohen et al. for orthogonal meshes for the first family [12] and for any mesh for the second family of edge elements of any order [13]. The extension to triangular and tetrahedral meshes was realized by Elmkies et al. [16, 17] but lead to efficient approximations up to the second-order elements.

Due to the storage of the stiffness matrix, even by using mass lumping techniques, FETD remained much more expensive than FDTD in terms of storage and, to a lesser extent, in computational time. This ultimate problem was solved by using a mixed $H(curl) - L^2$ formulation of Maxwell's equations based on $H(curl)$-conform definition of the *curl* operator in both spaces [13]. This technique provides a local definition of the stiffness matrices which induces a substantial gain of storage. It was later extended to acoustics [6] and linear elastodynamics [7]. A detailed presentation of all these techniques can be found in [5].

Unfortunately, although H^1 spectral elements and H^1 and $H(curl)$ triangular and tetrahedral elements behave quite well for any mesh, $H(curl)$ spectral elements present important parasitic waves for very distorted meshes, which are often used in industrial problems. For this reason, discontinuous Galerkin methods appeared as an efficient alternative for Maxwell's equations. First introduced by Hesthaven [21] for tetrahedra, this approach was adapted by Cohen et al. [9] to the spectral element point of view, which provided a low-storage as well as fast method to solve Maxwell's equations. This approach seemed to deal better with parasitic waves but eigenvalues considerations showed that such waves were however present in this method. All these remarks motivated us to discuss the numerical dissipation terms which can attenuate parasitic waves. This discussion is the new part of our survey.

Our paper is divided into four parts. In a first section, we present the continuous formulations of the Maxwell's equations and different approaches for its approximation. In the second section, we discuss the parasitic modes through an eigenvalue analysis. In the third section, we introduce dissipative jump terms to get rid of parasitic modes. The last section is devoted to the approximation of the time-harmonic problem by the methods described in the first section. Finally, some numerical experiments are presented.

2. Different Approximations of the Problem

2.1. Formulations of the continuous problem

In this paper, we are interested in solving the so-called lossy Maxwell's equations in anisotropic media which read