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RADIATION BOUNDARY CONDITIONS FOR MAXWELL'S EQUATIONS: A REVIEW OF ACCURATE TIME-DOMAIN FORMULATIONS *1)

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Abstract

We review time-domain formulations of radiation boundary conditions for Maxwell's equations, focusing on methods which can deliver arbitrary accuracy at acceptable computational cost. Examples include fast evaluations of nonlocal conditions on symmetric and general boundaries, methods based on identifying and evaluating equivalent sources, and local approximations such as the perfectly matched layer and sequences of local boundary conditions. Complexity estimates are derived to assess work and storage requirements as a function of wavelength and simulation time.

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1. Introduction

As the radiation of energy to the far field is an important feature of most problems in computational electromagnetics, an accurate and efficient truncation of the domain is a practical necessity for computations. In recent years there have been rapid developments in this field. In this review we will concentrate on strategies which can provide arbitrary accuracy. These include a variety of exact boundary condition formulations, which are all nonlocal in space and time, in addition to convergent local approximations such as the perfectly matched layer (PML). Besides describing the basic mathematical and algorithmic content of the various methods, we will, when possible, estimate their computational complexity as a function of the harmonic content of the field and the simulation time. Our goal is not to advocate one of the methods discussed over another. We will see that they are all capable of providing excellent accuracy at acceptable cost in many settings, and that an optimal choice will depend both on the details of the problem as well as on the time to be invested on code development.

We will assume that in the far field, that is beyond the computational domain Ω, we have a homogeneous, isotropic, dielectric material. In `cgs` units the source–free Maxwell equations

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then are:

$$\frac{\partial E}{\partial t} - c\nabla \times B = 0, \tag{1.1}$$

$$\frac{\partial B}{\partial t} + c\nabla \times E = 0, \tag{1.2}$$

subject to the constraints

$$\nabla \cdot E = 0 = \nabla \cdot B. \tag{1.3}$$

The constraints (1.3) are clearly preserved under the time evolution governed by (1.1)–(1.2).

Our problem is to specify radiation boundary conditions at an artificial boundary $\Gamma \subset \partial\Omega$ so that the solution computed in Ω can be made arbitrarily close to the restriction to Ω of the solution of the original problem on the unbounded domain. We will organize the discussion around four general classes of methods: fast methods based on separation of variables on symmetric boundaries, methods for general boundaries based on the retarded potential, methods based on equivalent source representations, and, finally, convergent local approximations including the perfectly matched layer (PML). We note that there have been parallel developments for other applications and refer the reader to [35, 36] for more comprehensive if slightly older reviews.

2. Boundaries with Symmetry

For planar, spherical, and cylindrical boundaries, this section formulates exact nonreflecting boundary conditions for the homogeneous Maxwell equations. An earlier review article [35] described these boundary conditions from a more general perspective. In contrast, our presentation here considers only the Maxwell system (1.1)–(1.2) and derives the relevant boundary conditions and effective numerical approximations from the ground up.

2.1. Planar boundary

Let $x_1 = x = 0$ specify the planar boundary of the "computational domain" $x < 0$. On the system (1.1)–(1.2) we perform both a Laplace transform (denoted by a hat) in time and a Fourier transform (denoted by a bar) in the tangential variables $(x_2, x_3) = (y, z)$, thereby obtaining a differential–algebraic system. With (k_2, k_3) representing the Fourier variables dual to (y, z), the system's algebraic sector is

$$\tilde{s}\hat{\bar{E}}_1 = ik_2\hat{\bar{B}}_3 - ik_3\hat{\bar{B}}_2, \qquad \tilde{s}\hat{\bar{B}}_1 = -ik_2\hat{\bar{E}}_3 + ik_3\hat{\bar{E}}_2, \tag{2.1}$$

where $\tilde{s} = s/c$. Using these algebraic equations, we may then express the remaining differential sector solely in terms of the tangential variables as follows:

$$\frac{\partial}{\partial x}\begin{pmatrix} \hat{\bar{E}}_2 \\ \hat{\bar{E}}_3 \\ \hat{\bar{B}}_2 \\ \hat{\bar{B}}_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & k_2k_3/\tilde{s} & -(\tilde{s}^2 + k_2^2)/\tilde{s} \\ 0 & 0 & (\tilde{s}^2 + k_3^2)/\tilde{s} & -k_2k_3/\tilde{s} \\ -k_2k_3/\tilde{s} & (\tilde{s}^2 + k_2^2)/\tilde{s} & 0 & 0 \\ -(\tilde{s}^2 + k_3^2)/\tilde{s} & k_2k_3/\tilde{s} & 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{\bar{E}}_2 \\ \hat{\bar{E}}_3 \\ \hat{\bar{B}}_2 \\ \hat{\bar{B}}_3 \end{pmatrix}. \tag{2.2}$$

The eigenvalues of the matrix are

$$\lambda_\pm = \pm\sqrt{\tilde{s}^2 + k_2^2 + k_3^2} = \pm\sqrt{\tilde{s}^2 + |k|^2}, \tag{2.3}$$