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SURFACE FINITE ELEMENTS FOR PARABOLIC EQUATIONS *1)

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Abstract

In this article we define a surface finite element method (SFEM) for the numerical solution of parabolic partial differential equations on hypersurfaces Γ in $\mathbb{R}^{n+1}$. The key idea is based on the approximation of Γ by a polyhedral surface Γ_h consisting of a union of simplices (triangles for $n = 2$, intervals for $n = 1$) with vertices on Γ. A finite element space of functions is then defined by taking the continuous functions on Γ_h which are linear affine on each simplex of the polygonal surface. We use surface gradients to define weak forms of elliptic operators and naturally generate weak formulations of elliptic and parabolic equations on Γ. Our finite element method is applied to weak forms of the equations. The computation of the mass and element stiffness matrices are simple and straightforward. We give an example of error bounds in the case of semi-discretization in space for a fourth order linear problem. Numerical experiments are described for several linear and nonlinear partial differential equations. In particular the power of the method is demonstrated by employing it to solve highly nonlinear second and fourth order problems such as surface Allen-Cahn and Cahn-Hilliard equations and surface level set equations for geodesic mean curvature flow.

Mathematics subject classification: 65M60, 65M30, 65M12, 65Z05, 58J35, 53A05, 74S05, 80M10, 76M10.
Key words: Surface partial differential equations, Surface finite element method, Geodesic curvature, Triangulated surface.

1. Introduction

Partial differential equations on surfaces occur in many applications. For example, traditionally they arise naturally in fluid dynamics and material science and more recently in the mathematics of images. In this paper we propose a mathematical approach to the formulation and finite element approximation of parabolic equations on a surface in $\mathbb{R}^{n+1}$ $(n = 1, 2)$. We give examples of linear and nonlinear equations. In particular we show how surface level set and phase field models can be used to compute the motion of curves on surfaces.

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1.1. The diffusion equation

Conservation on a hypersurface Γ of a scalar u with a diffusive flux $-\mathcal{D}\nabla_\Gamma w$, where $\mathcal{D}$ is the diffusivity tensor and w is a scalar, leads to the diffusion equation

$$u_t - \nabla_\Gamma \cdot (\mathcal{D}\nabla_\Gamma w) = 0 \tag{1.1}$$

on Γ. Here ∇_Γ is the tangential or surface gradient. If $\partial\Gamma$ is empty then the equation does not need a boundary condition. Otherwise we can impose Dirichlet or no flux boundary conditions on $\partial\Gamma$. Choosing various constitutive relations to define the relationship between the flux and u leads to a variety of second and fourth order linear and nonlinear parabolic equations. For example the constitutive relations $w = u$ and $w = -\Delta_\Gamma u$ lead to linear second and fourth order diffusion equations.

1.2. The finite element method

In this paper we propose a finite element approximation based on the variational form

$$\int_\Gamma u_t\varphi + \int_\Gamma \mathcal{D}\nabla_\Gamma w \cdot \nabla_\Gamma \varphi = 0 \tag{1.2}$$

where φ is an arbitrary test function defined on the surface Γ in $\mathbb{R}^3$ with $\partial\Gamma$ empty. This provides the basis of our surface finite element method (SFEM) which is applicable to arbitrary n–dimensional hypersurfaces in $\mathbb{R}^{n+1}$ (curves in $\mathbb{R}^2$) with or without boundary. Indeed this is the extension of the method from [10] for the Laplace-Beltrami equation, which was extended to linear second order diffusion equations on moving surfaces in [12]. We focus our description on the case $n = 2$ but observe that the approach is directly applicable to $n = 1$.

The principal idea is to use a polyhedral approximation of Γ based on a triangulated surface. It follows that a quite natural local piecewise linear parametrization of the surface is employed rather than a global one. The finite element space is then the space of continuous piecewise linear functions on the triangulated surface. The implementation is thus rather similar to that for solving the diffusion equation on flat stationary domains. For example, for $w = u$, the backward Euler time discretization leads to the SFEM scheme

$$\frac{1}{\tau}\left(\mathcal{M}\alpha^{m+1} - \mathcal{M}\alpha^m\right) + \mathcal{S}\alpha^{m+1} = 0$$

where $\mathcal{M}$ and $\mathcal{S}$ are the surface mass and stiffness matrices and α^m is the vector of nodal values for the approximation of u at time t^m. Here, τ denotes the time step size. Observe that this approach to evolutionary surface partial differential equations was used in [11] to evolve a surface by mean curvature flow. See also [5].

1.3. Level set or implicit surface approach

An alternative approach to our method based on the use of (1.2) is to embed the surface in a family of level set surfaces [1, 3, 4, 13, 14, 21, 30]. This Eulerian approach can be discretized on a Cartesian grid in $\mathbb{R}^{n+1}$ and has the usual advantages and disadvantages of level set methods. Equations on surfaces also arise in phase field models [7, 19, 25].