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SOME n-RECTANGLE NONCONFORMING ELEMENTS FOR FOURTH ORDER ELLIPTIC EQUATIONS *1)

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Abstract

In this paper, three n-rectangle nonconforming elements are proposed with $n \geq 3$. They are the extensions of well-known Morley element, Adini element and Bogner-Fox-Schmit element in two spatial dimensions to any higher dimensions respectively. These elements are all proved to be convergent for a model biharmonic equation in n dimensions.

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1. Introduction

Motivated by both theoretical and practical interests, we will consider n-rectangle ($n \geq 2$) nonconforming finite elements for n-dimensional fourth order partial equations in this paper. In the two dimensional case, there are well-known nonconforming elements, such as the Morley element, the Zienkiewicz element and the Adini element, etc (see [1-4]). In a recent paper [10], we have discussed the motivation to construct nonconforming finite elements in three dimensions and proposed some tetrahedral nonconforming finite elements for 3-dimensional fourth order partial equations. As for the Morley element, we have extended it to any higher simplex case in another paper [11].

In this paper, we extend the Morley element, the Adini element and the Bogner-Fox-Schmit element to any higher dimensions, and obtain the following three types of n-rectangle nonconforming finite elements:

1. The n-rectangle Morley element, whose degrees of freedom are the value of the normal derivative at the centric point of each $(n-1)$-dimensional face and the function value at each vertex.

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2. The n-dimensional Adini element, whose degrees of freedom are the values of function and all first order derivatives at each vertex.

3. The n-dimensional BFS element, whose degrees of freedom are the values of function, all first order derivatives and all second order mixed derivatives at each vertex.

We will use the following standard notation. Ω denotes a general bounded polyhedral domain in R^n $(n \geq 2)$, $\partial\Omega$ the boundary of Ω, and $\nu = (\nu_1, \nu_2, \cdots, \nu_n)^\top$ the unit outer normal to $\partial\Omega$. For a nonnegative integer s, $H^s(\Omega)$, $H_0^s(\Omega)$, $\|\cdot\|_{s,\Omega}$ and $|\cdot|_{s,\Omega}$ denote the usual Sobolev spaces, its corresponding norm and semi-norm respectively, and $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$.

Given a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, set $|\alpha| = \sum_{i=1}^n \alpha_i$ and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\forall x \in R^n$. For a subset $B \subset R^n$ and a nonnegative integer r, let $P_r(B)$ and $Q_r(B)$ be the spaces of polynomials on B defined by

$$P_r(B) = \mathrm{span}\{\, x^\alpha \mid |\alpha| \leq r\}, \quad Q_r(B) = \mathrm{span}\{\, x^\alpha \mid \alpha_i \leq r\}.$$

The paper is organized as follows. The rest of this section gives some notation. Section 2 gives a detailed description of the n-rectangle Morley element, the n-dimensional Adini element and the BFS element. Section 3 and Section 4 show the convergence of these elements.

2. The n-Rectangle Elements

In this section, we will give our extensions of the Morley element, the Adini element and the Bogner-Fox-Schmit element to higher dimensions. For a finite element, it can be described by a triple (T, P_T, Φ_T) with T the geometric shape, P_T the shape function space and Φ_T the vector of degrees of freedom.

Given $a_0 = (a_{01}, a_{02}, \cdots, a_{0n})^\top \in R^n$ and positive numbers $h_1, \cdots, h_n$, an n-rectangle T is given by

$$T = \{\, x \,|\, x_i = a_{0i} + h_i\xi_i,\ -1 \leq \xi_i \leq 1,\ 1 \leq i \leq n\}.$$

Let $\xi = (\xi_1, \cdots, \xi_n)^\top$, and let a_i, $1 \leq i \leq 2^n$, be the vertices of T. The vertices are written by

$$a_i = (a_{01} + \xi_{i1}h_1,\ a_{02} + \xi_{i2}h_2, \cdots, a_{0n} + \xi_{in}h_n)^\top, \quad 1 \leq i \leq 2^n,$$

and the barycenters of the $(n-1)$-dimensional faces of T are written as

$$\begin{cases} b_{2k-1} = (a_{01}, \cdots, a_{0,k-1}, a_{0k} + h_k, a_{0,k+1}, \cdots, a_{0n})^\top, \\ b_{2k} \quad\ = (a_{01}, \cdots, a_{0,k-1}, a_{0k} - h_k, a_{0,k+1}, \cdots, a_{0n})^\top, \end{cases} \quad 1 \leq k \leq n.$$

Let F_i $(1 \leq i \leq 2n)$ denote the $(n-1)$-dimensional face with b_i as its barycenter. Define

$$\tilde{p}_i = \frac{1}{2^n} \prod_{j=1}^n (1 + \xi_{ij}\xi_j), \quad 1 \leq i \leq 2^n.$$

It is known that $\tilde{p}_i$, $1 \leq i \leq 2^n$, forms a basis of $Q_1(T)$. For a mesh size h, let $\mathcal{T}_h$ be a triangulation of Ω consisting of n-rectangles described above.