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ESTIMATING ERROR BOUNDS FOR TERNARY SUBDIVISION CURVES/SURFACES *1)

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Abstract

We estimate error bounds between ternary subdivision curves/surfaces and their control polygons after k-fold subdivision in terms of the maximal differences of the initial control point sequences and constants that depend on the subdivision mask. The bound is independent of the process of subdivision and can be evaluated without recursive subdivision. Our technique is independent of parametrization therefore it can be easily and efficiently implemented. This is useful and important for pre-computing the error bounds of subdivision curves/surfaces in advance in many engineering applications such as surface/surface intersection, mesh generation, NC machining, surface rendering and so on.

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1. Introduction

Subdivision is an important method for generating smooth curves and surfaces, see, e.g., [1, 2, 8]. Efficiency of subdivision algorithms, their flexibility and simplicity have found their way into wide applications in Computer Graphics and Computer Aided Geometric Design (CAGD). A widely used, efficient and intuitive way to specify, represent and reason about curved, surfaces, nonlinear geometry for design and modeling is the control polygon paradigm. For many applications, e.g., rendering, intersection testing or design, this raises the question just how well the control polygon approximates the exact curved and surface geometry. Several researchers give several answers to this question. Nairn et al. [7] show that the maximal distance between a Bézier segment and its control polygon is bounded in terms of the differences of the control point sequence and a constant that depends only on the degree of the polynomial. Lutterkort and Peters [6] derived a sharp bound on the distance between a spline and its B-spline control polygon. Their bound yields a piecewise linear envelope enclosing the spline and the control polygon. Recently, Karavelas et al. [5] derived sharp bounds for the distance between a planar parametric Bézier curve and parameterizations of its control polygon based on the Greville abscissae. In [1], Cheng gave an algorithm to estimate subdivision depths for rational curves and surfaces. The subdivision depth is not estimated for the given curve/surface

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directly. Their algorithm computes a subdivision depth for the polynomial curve/surface of which the given rational curve/surface is the image under the standard perspective projection. The existing methods for computing the bounds on the approximation of polynomials and splines by their control structures are all based on the parameterizations, so that it is very difficult for them to be generalized to the subdivision surfaces.

In this paper, we estimate error bounds for ternary subdivision curves/surfaces in terms of the maximal differences of the initial control point sequence and constants that depend on the subdivision mask. Our technique is independent of parameterizations and therefore it can be easily and efficiently implemented. The paper is organized as follows.

In Section 2 we prove the first main result of the paper about the estimation of error bounds between ternary subdivision curves and their control polygon after k-fold subdivision. Then as an application of our result we find error bounds for 3-point ternary approximating [3], 3-point ternary interpolatory [3] and 4-point ternary interpolatory [4] subdivision schemes. In Section 3 we generalize the main result of Section 2 to estimate the error bounds between subdivision surfaces and their control polygons. In Section 4, we summarize the results obtained and make some comments for future research directions.

2. The Error Bounds for Ternary Subdivision Curves

Let $p_i^k \in \mathbb{R}^N$, $i \in \mathbb{Z}$, denote a sequence of points in $\mathbb{R}^N$, $N \geqslant 2$, where k is a nonnegative integer. A ternary subdivision process [3] is defined by

$$p_{3i+s}^{k+1} = \sum_{j=0}^{m} a_{s,j} p_{i+j}^k, \quad s = 0, 1, 2, \tag{2.1}$$

where $m > 0$ and

$$\sum_{j=0}^{m} a_{s,j} = 1, \quad s = 0, 1, 2. \tag{2.2}$$

The coefficients $\{a_{s,j}\}_{j=0}^m$, $0 \leq s \leq 2$, are called subdivision mask. Given initial values $p_i^0 \in \mathbb{R}^N, i \in \mathbb{Z}$. Then in the limit $k \to \infty$, the process defines an infinite set of points in $\mathbb{R}^N$. The sequence of control points $\{p_i^k\}$ is related, in a natural way, with the diadic mesh points $t_i^k = i/3^k$, $i \in \mathbb{Z}$. The process (2.1) then defines a scheme whereby p_{3i}^{k+1} replaces the value p_i^k at the mesh point $t_{3i}^{k+1} = t_i^k$ and p_{3i+1}^{k+1} and p_{3i+2}^{k+1} are inserted at the new mesh points $t_{3i+1}^{k+1} = \frac{1}{3}(2t_i^k + t_{i+1}^k)$ and $t_{3i+2}^{k+1} = \frac{1}{3}(t_i^k + 2t_{i+1}^k)$ respectively.

We now establish our first main result for error bounds between subdivision curves and their control polygons.

Theorem 2.1. *Given initial control polygon $p_i^0 = p_i$, $i \in \mathbb{Z}$, and let the values p_i^k, $k \geqslant 0$ be defined recursively by subdivision process* (2.1) *together with* (2.2)*. Suppose P^k be the piecewise linear interpolant to the values p_i^k and P^∞ be the limit curve of the process* (2.1)*. If*

$$\delta = \max\left\{\sum_{j=0}^{m} |d_j|, \sum_{j=0}^{m} |e_j|, \sum_{j=0}^{m} |f_j|\right\} < 1, \tag{2.3}$$

where

$$d_j = \sum_{t=0}^{j} (a_{0,t} - a_{1,t}), \quad e_j = \sum_{t=0}^{j} (a_{1,t} - a_{2,t}), \quad f_j = a_{0,j} - (d_j + e_j), \tag{2.4}$$