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MODIFIED BERNOULLI ITERATION METHODS FOR QUADRATIC MATRIX EQUATION *

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Abstract

We construct a modified Bernoulli iteration method for solving the quadratic matrix equation $AX^2 + BX + C = 0$, where A, B and C are square matrices. This method is motivated from the Gauss-Seidel iteration for solving linear systems and the Sherman-Morrison-Woodbury formula for updating matrices. Under suitable conditions, we prove the local linear convergence of the new method. An algorithm is presented to find the solution of the quadratic matrix equation and some numerical results are given to show the feasibility and the effectiveness of the algorithm. In addition, we also describe and analyze the block version of the modified Bernoulli iteration method.

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Key words: Quadratic matrix equation, Quadratic eigenvalue problem, Solvent, Bernoulli's iteration, Newton's method, Local convergence.

1. Introduction

Quadratic matrix equations (**QME**) appear in many areas of scientific computing and engineering applications. Besides the famous *algebraic Riccati equation* (**ARE**) in control theory [5, 12, 18], the quadratic matrix equation

$$\mathcal{Q}(X) = AX^2 + BX + C = 0, \quad \text{with} \quad A,\ B,\ C \in \mathbb{R}^{n\times n}, \tag{1.1}$$

occurs in a variety of problems. For example, the *quadratic eigenvalue problem* (**QEP**) arising in the analysis of damped structural systems and vibration problems [6, 7, 15, 17, 26, 27], the *Quasi-Birth-Death* (**QBD**) problem used as stochastic models in telecommunication computer performance and inventory control [4, 19], and the noisy Wiener-Hopf problems coming from Markov chains [10, 16, 24, 25].

A lot of work has been done for computing the numerical solution of the QME (1.1). Davis [6] considered Newton's method, Higham and Kim [15] studied Newton's method with exact line search, and He, Beini and Rhee [13] discussed a cyclic reduction algorithm for the QBD problem; see also [3]. Besides, as special cases of the quadratic matrix equation (1.1), several linear matrix equations have been studied in [20, 21, 22, 8] where existence conditions and direct algorithms about the solutions were presented. Other different techniques for analyzing and computing the solutions of some general matrix equations can be found in [1, 11].

From the discussion in [14] we know that the *Bernoulli iteration* (**BI**) method is more efficient for over-damped quadratic eigenvalue problems and the QBD problem. This motivates

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us to further study this iteration method for the QME (1.1). By making use of techniques of Gauss-Seidel relaxation sweep [9] for solving linear systems and the Sherman-Morrison-Woodbury formula [2, 9] for updating matrix, we establish a new iteration method and its block version for solving the QME (1.1). These methods are technical modifications of the Bernoulli iteration method in [14], and are of lower computing costs in actual applications. Convergence analyses show that these new methods have local linear convergence rates, and numerical implementations show that they are feasible and effective solvers for the QME (1.1).

The organization of the paper is as follows. After introducing some basic notations and concepts in Section 2, we establish the general *modified Bernoulli iteration* (**MBI**) method in Section 3. In Section 4, we derive a block version of the MBI method, which is called as *block modified Bernoulli iteration* (**BMBI**) method. In Section 5, we study the local convergence properties of both MBI and BMBI methods under suitable conditions. In Section 6, some numerical results are given to show the feasibility and effectiveness of our new methods. Finally, in Section 7, we use some remarks to end this paper.

2. Notations and Concepts

We introduce some notations and concepts, which are necessary for our subsequent statements.

A solution of the QME (1.1) is also called a solvent. The QEP corresponding to the QME (1.1) is defined as

$$\mathcal{Q}(\lambda)x = (\lambda^2 A + \lambda B + C)x = 0. \tag{2.1}$$

As is known, if A is nonsingular, then $\mathcal{Q}(\lambda)$ has exactly $2n$ eigenvalues λ_j $(j = 1, 2, \cdots, 2n)$, which can be ordered with respect to their absolute values in the following form:

$$|\lambda_1| \geqslant |\lambda_2| \geqslant \cdots \geqslant |\lambda_{2n}|. \tag{2.2}$$

We use $\lambda(A)$ to denote the eigenvalue set of the matrix A.

Definition 2.1. ([14]) *Let the eigenvalues λ_j $(j = 1, 2, \ldots, 2n)$ of the QEP* (2.1) *be ordered in the form of* (2.2). *Then a solvent S_1 of Q(X) is called a dominant solvent if $\lambda(S_1) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and $|\lambda_n| > |\lambda_{n+1}|$; and a solvent S_2 of Q(X) is called a minimal solvent if $\lambda(S_2) = \{\lambda_{n+1}, \lambda_{n+2}, \ldots, \lambda_{2n}\}$ and $|\lambda_n| > |\lambda_{n+1}|$.*

One type of Bernoulli iteration methods for finding the dominant and the minimal solvents are as follows:

$$A + (B + CW_{k-1})W_k = 0, \qquad W_0 = 0,\ k = 1, 2, \ldots \tag{2.3}$$

and

$$(AX_{k-1} + B)X_k + C = 0, \qquad X_0 = 0,\ k = 1, 2, \ldots. \tag{2.4}$$

From [14, Theorem 10], we see that

$$\lim_{k\to\infty} W_k = S_1^{-1} \quad \text{and} \quad \lim_{k\to\infty} X_k = S_2$$

hold under the condition that the QME (1.1) has a dominant solvent S_1 and a minimal solvent S_2. Moreover, the asymptotic convergence rates of these two iterations are linear, with the convergence factor $\sigma = |\lambda_{n+1}|/|\lambda_n|$.