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A QR DECOMPOSITION BASED SOLVER FOR THE LEAST SQUARES PROBLEMS FROM THE MINIMAL RESIDUAL METHOD FOR THE SYLVESTER EQUATION *

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Abstract

Based on the generalized minimal residual (GMRES) principle, Hu and Reichel proposed a minimal residual algorithm for the Sylvester equation. The algorithm requires the solution of a structured least squares problem. They form the normal equations of the least squares problem and then solve it by a direct solver, so it is susceptible to instability. In this paper, by exploiting the special structure of the least squares problem and working on the problem directly, a numerically stable QR decomposition based algorithm is presented for the problem. The new algorithm is more stable than the normal equations algorithm of Hu and Reichel. Numerical experiments are reported to confirm the superior stability of the new algorithm.

Mathematics subject classification: 65F22, 65K10.
Key words: Least-squares solution, Preconditioning, Generalized singular value decomposition.

1. Introduction

Consider the numerical solution of the Sylvester equation

$$AX - XB = C, \tag{1.1}$$

where $A \in R^{n_1 \times n_1}, B \in R^{n_2 \times n_2}$ and $C \in R^{n_1 \times n_2}$ are given matrices, and X is the solution matrix. Such kind of problems arise in various settings, and there are many methods for solving them [1, 2, 3, 4, 9, 10]. For large and sparse problems, based on the GMRES algorithm [8] for large unsymmetric linear systems, Hu and Reichel [6] present a minimal residual Krylov subspace method. In this case, a least squares problem of Kronecker product form must be solved, and a similar problem also sees [7]. In [6] they first form the normal equations system of the least squares problem and then solve it by a direct solver. Their algorithm is susceptible to instability and the computed solution may have poor accuracy due to the possibly high ill-conditioning of the normal equations system. In this paper we propose a new algorithm for solving the least squares problem. It is based on stable QR decompositions and fully exploit the special structure of the problem. Thus, it is more stable than a normal equations based solver. We also compare the cost of our algorithm with that of Hu and Reichel, showing that ours is a little bit more expensive than theirs but both are negligible, compared to the overall cost of the minimal residual method. This indicates that our improvement is significant.

In Section 2 we review the least squares problem to be solved in the minimal residual method. In Section 3 we show how to develop a stable QR decomposition based algorithm

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for the resulting special least squares problem. Meanwhile, we compare the costs of our new algorithm and theirs. In Section 4 we report some numerical examples to show the stability and higher accuracy of our algorithm.

Some notations to be used in the paper are introduced. Denote by the superscript T the transpose of a vector or a matrix, by A^+ the generalized inverse of A, by $\|\cdot\|_2$ the spectral norm, by $\kappa_2(A) = \|A\|_2\|A^+\|_2$ the condition number of A, by $\mathcal{R}^m$ the set of m-dimensional vectors, by $\mathcal{R}^{m\times n}$ the set of $m \times n$ matrices. Throughout the paper, I is an identity with order clear in the context, e_i is the ith row of I, and $\tilde{I}$ is the same as the identity matrix I with an additional zero row.

2. The Least Squares Problem

The minimal residual method [6] given by Hu and Reichel replaces the subspace $\mathcal{K}_m(I \otimes A - B^T \otimes I, r_0)$ by another subspace of the form $\mathcal{K}_m(B^T, g) \otimes \mathcal{K}_m(A, f)$ for certain vectors $f, g \in R^n$. It is seen from Algorithm 5.2 of [6] that the construction of f, g needs $\mathcal{O}(n_1 n_2)$ flops. We should point out that usually the two subspaces are different. Hu and Reichel then use the Arnoldi process to generate the Hessenberg matrices $\tilde{H}_A$, $\tilde{H}_B$ and the orthonormal bases W_m and V_m of $\mathcal{K}_m(B^T, g)$ and $\mathcal{K}_m(A, f)$, respectively. The process uses $2n_1m^2$ flops + m matrix A by vector products and $2n_2m^2$ flops + m matrix B^T by vector products. Then one solves a least squares problem of the form

$$\min_{y_m \in \mathcal{R}^{m^2}} \|\tilde{r}_0 - (\tilde{I} \otimes \tilde{H}_A - \tilde{H}_B \otimes \tilde{I})y_m\|_2, \tag{2.1}$$

where $\tilde{I}$ is defined as before and $\tilde{H}_A, \tilde{H}_B$ are $(m+1) \times m$ Hessenberg matrices. Assume

$$H_A = Q_A R_A Q_A^T, \quad H_B = Q_B R_B Q_B^T$$

to be the Schur decompositions of H_A and H_B, where H_A and H_B are the first m rows of $\tilde{H}_A$ and $\tilde{H}_B$, and define the unitary matrices

$$\tilde{Q}_A = \begin{pmatrix} Q_A & \\ & 1 \end{pmatrix}, \quad \tilde{Q}_B = \begin{pmatrix} Q_B & \\ & 1 \end{pmatrix}$$

and the matrices

$$\tilde{R}_A = \tilde{Q}_A^T \tilde{H}_A Q_A, \quad \tilde{R}_B = \tilde{Q}_B^T \tilde{H}_B Q_B.$$

Then the leading $m \times m$ principal submatrix of $\tilde{R}_A$ is R_A, and the $(m+1)$st row of $\tilde{R}_A$ is given by

$$e_{m+1}^T \tilde{R}_A e_j = e_{m+1}^T \tilde{H}_A e_m e_m^T Q_A e_j, \quad 1 \le j \le m.$$

A similar result holds for $\tilde{R}_B$.

Let $r_0' = (\tilde{Q}_B \otimes \tilde{Q}_A)^T \tilde{r}_0$ and $y_m' = (Q_B \otimes Q_A)^T y_m$. Then (2.1) is equivalent to

$$\min_{y_m \in R^{m^2}} \|r_0' - (I_{m+1,m} \otimes \tilde{R}_A - \tilde{R}_B \otimes I_{m+1,m})y_m'\|_2. \tag{2.2}$$

The $(m+1)^2 \times m^2$ matrix in (2.2) has m^2 rows in common with the upper triangular matrix $R = (I \otimes R_A - R_B \otimes I)$. Let the remaining rows define the $(2m+1) \times m^2$ matrix S. Thus, for