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INCREMENTAL UNKNOWNS FOR THE HEAT EQUATION WITH TIME-DEPENDENT COEFFICIENTS: SEMI-IMPLICIT θ-SCHEMES AND THEIR STABILITY *

Yu-Jiang Wu and Ai-Li Yang
(*School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China*
Email: myjaw@lzu.edu.cn)

Abstract

Based on the finite difference discretization of partial differential equations, we propose a kind of semi-implicit θ-schemes of incremental unknowns type for the heat equation with time-dependent coefficients. The stability of the new schemes is carefully studied. Some new types of conditions give better stability when θ is closed to 1/2 even if we have variable coefficients.

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Key words: Incremental unknowns, Semi-implicit schemes, θ-Schemes, Stability.

1. Introduction

Using finite difference discretization in the infinite dimensional dynamical systems to seek the solution of nonlinear partial differential equations and to study its long time behavior is highly stressed by many authors, see for example [4,10,12]. The Incremental Unknowns (IU) method, stemming originally from the dynamical system theory, was introduced by Temam in 1990 ([11]) for the approximation of inertial manifolds when finite differences are used to discretize a partial differential equation, see also [5,12]. It was shown that the IU method usually yields a very well conditioned matrix in the IU-type linear algebraic equations. Many articles have contributed to the analysis of the IU method and to applying the property to several kinds of differential equations.

For the heat equation of constant coefficients, Pouit [8] constructed a Y-explicit and Z-implicit IU-scheme, and Huang and Wu [7] constructed a class of weighted IU-schemes. The objective of this work is to construct a new type of semi-implicit θ-schemes for the heat equation with time-dependent coefficients which are monotonous increasing with respect to time. We will study the stability of the new schemes and give the proof of the stability theorem.

2. Semi-Implicit θ-Schemes

We consider the one-dimensional evolution equation, i.e., the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - v(t)\dfrac{\partial^2 u}{\partial x^2} = f, & 0 < x < 1, \quad 0 < t \le T, \\ u(0,t) = u(1,t) = 0, & 0 \le t \le T, \\ u(x,0) = u_0, & 0 < x < 1. \end{cases} \tag{2.1}$$

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where $v(t)$ expresses the time-dependent coefficients when t varies in $[0,T]$. Suppose that $v(t)$ is increasing and bounded on $[0,T]$ with $v(0)>0$.

At first, we discretize (2.1) by finite differences. By introducing the incremental unknowns U^m and the transfer matrix S (see, [4,7,8,11])

$$U^m=\begin{pmatrix} Y^m \\ Z^m \end{pmatrix}, \qquad S=\begin{pmatrix} I_{N-1} & 0 \\ G & I_N \end{pmatrix},$$

where the $N\times(N-1)$ matrix $G=(g_{ij})$ is given by $g_{ij}=0$ except that $g_{ii}=g_{i+1,i}=\frac{1}{2}$, we can construct the IU-type θ-scheme ([6,9,13])

$$\begin{aligned}&\big(S^TS+\theta\Delta t v^m S^TAS\big)\begin{pmatrix} Y^m \\ Z^m \end{pmatrix}\\ &=\big(S^TS-(1-\theta)\Delta t v^m S^TAS\big)\begin{pmatrix} Y^{m-1} \\ Z^{m-1} \end{pmatrix}+\Delta t S^TS\begin{pmatrix} F_Y^m \\ F_Z^m \end{pmatrix}.\end{aligned}\tag{2.2}$$

If we have the basis $(\varphi_p)_{p=1,2,\cdots,2N-1}$ in $\mathbb{R}^{2N-1}$, then the scheme (2.2) becomes

$$\sum_{p=1}^{2N-1}(1+\theta\Delta t v^m\lambda_p)\mathcal{U}_p^m\varphi_p=\sum_{p=1}^{2N-1}[(1-(1-\theta)\Delta t v^m\lambda_p)\mathcal{U}_p^{m-1}+\Delta t\mathcal{F}_p^m]\varphi_p.$$

Consequently, the solution of (2.2) can be written as

$$\mathcal{U}_p^m=\frac{1-(1-\theta)\Delta t v^m\lambda_p}{1+\theta\Delta t v^m\lambda_p}\mathcal{U}_p^{m-1}+\frac{\Delta t}{1+\theta\Delta t v^m\lambda_p}\mathcal{F}_p^m.$$

It is easy to show that

$$\frac{1-(1-\theta)\Delta t v^m\lambda_p}{1+\theta\Delta t v^m\lambda_p}<1$$

can be satisfied unconditionally. Note that since $\lambda_1<\lambda_2<\cdots<\lambda_{2N-1}$ and $\lambda_{2N-1}\sim\frac{4}{h^2}, (N\to\infty)$, we get the stability condition of (2.2).

Proposition 2.1. *The stability condition of the IU-type θ-scheme* (2.2) *is as follows:*

1) $0\le\theta<\frac{1}{2}$, $\quad \Delta t<\dfrac{h^2}{2(1-2\theta)v(T)}$,
2) $\theta=\frac{1}{2}$, *unconditionally stable* (it is the Crank-Nicolson scheme),
3) $\frac{1}{2}<\theta\le 1$, *unconditionally stable.*

In terms of the incremental unknowns, the semi-implicit θ-schemes of (2.1) can be written as

$$\begin{aligned}&S^TS\begin{pmatrix} Y^m \\ Z^m \end{pmatrix}+\theta v^m\Delta t S^TAS\begin{pmatrix} Y^{m-1} \\ Z^m \end{pmatrix}\\ &=(S^TS-(1-\theta)v^m\Delta t S^TAS)\begin{pmatrix} Y^{m-1} \\ Z^{m-1} \end{pmatrix}+\Delta t S^TS\begin{pmatrix} F_Y^m \\ F_Z^m \end{pmatrix}.\end{aligned}\tag{2.3}$$

Since

$$S^TS=\begin{pmatrix} B & G^T \\ G & I_N \end{pmatrix}, \qquad S^TAS=\begin{pmatrix} \frac{A^*}{2} & 0 \\ 0 & \frac{2}{h^2}I_N \end{pmatrix},\tag{2.4}$$