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FIRST-ORDER METHODS FOR SOLVING THE OPTIMAL STATIC $\mathcal{H}_\infty$-SYNTHESIS PROBLEM *

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Abstract

In this paper, we consider the static output feedback (SOF) $\mathcal{H}_\infty$-synthesis problem posed as a nonlinear semi-definite programming (NSDP) problem. Two numerical algorithms are developed to tackle the NSDP problem by solving the corresponding Karush-Kuhn-Tucker first-order necessary optimality conditions iteratively. Numerical results for various benchmark problems illustrating the performance of the proposed methods are given.

Mathematics subject classification: 49N35, 49N10, 93D52, 93D22, 65K05.
Key words: Static output feedback, $\mathcal{H}_\infty$-synthesis, Semi-definite programming, Nonlinear programming.

1. Introduction

In this paper, we consider the following NSDP problem:

$$(\mathrm{P0}): \begin{cases} \min\limits_{X,V,\gamma} & \gamma \\ \text{s.t.} & H(X,\gamma)=0, \quad Y(X,V,\gamma)\prec 0, \quad V\succ 0, \end{cases} \tag{1.1}$$

where $\gamma \in \mathbb{R}_+$, and $H:\mathbb{R}^{r\times t}\times\mathcal{S}^n\times\mathbb{R}_+\to\mathcal{S}^n$, $Y:\mathbb{R}^{r\times t}\times\mathcal{S}^n\times\mathcal{S}^n\times\mathbb{R}_+\to\mathcal{S}^n$ are assumed to be sufficiently smooth matrix functions. In the considered optimal control applications the variable X is a decomposition of a matrix pair $(F,L)\in\mathbb{R}^{r\times t}\times\mathcal{S}^n$, where $\mathcal{S}^n$ denotes the set of real symmetric $n\times n$ matrices. The problem (1.1) is a nonlinear semi-definite programming and is generally non-convex.

In recent years there were several attempts to employ the available successful computational techniques in nonlinear optimization to solve various NSDP problems numerically. Such NSDP problems represent a variety of applications in system and control theory; see among others [6, 7, 9, 12, 14, 17, 18, 19, 22, 24, 32]. In particular, the above NSDP problem (1.1) represents a wide range of applications in system and control theory; see, e.g., the benchmark collection COMPl$_\mathrm{e}$ib [25].

By solving (1.1) we mean computing a feasible point (X,V,γ) that satisfies the set of equality and inequality constraints as well as enforcing the objective γ to attain its least possible value. Due to the difficulties in solving problem (1.1) numerically, however, in most of the above citations the attempts were only to compute suboptimal solution.

The main goal of this paper is to propose two first-order methods for solving (1.1) that take advantage of the problem structure and use inexact computations. In these methods we

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compute a stationary point of the above optimization problem by solving numerically the nonlinear system of equations resulting from the Karush-Kuhn-Tucker (KKT) first-order necessary optimality conditions. It is particularly important in the proposed methods to deal with a scalar objective function together with matrix variables.

This paper is organized as follows. In the next section we present the problem formulation and then we state the assumptions imposed on the problem (1.1). In addition, we obtain the nonlinear system of equations resulting from the first-order necessary optimality conditions of the problem (1.1). In Section 3 we develop two first-order methods for computing approximately a stationary point of the problem (1.1). In Section 4 we test numerically the proposed algorithms through several test problems from the benchmark collection COMPl$_\text{e}$ib [25].

Notations: For a matrix $M \in \mathbb{R}^{n\times n}$ the notations $M \succ 0$, $M \prec 0$ denote that M is positive definite, negative definite, respectively. Throughout the paper, the symbol $\|\cdot\|$ denotes the Frobenius norm defined by $\|M\| = \sqrt{\langle M, M\rangle}$, where $\langle\cdot,\cdot\rangle$ is the inner product given by $\langle M_1, M_2\rangle = Tr\,(M_1^T M_2)$, and $Tr\,(\cdot)$ is the trace operator.

2. Problem Formulation

The solution of the $\mathcal{H}_\infty$ synthesis problem has received considerable attention in the control literature; see, e.g., [1-3, 6-10, 12, 14, 15, 17-19, 22, 24, 31, 32] and the references therein. Given a linear time-invariant (LTI) control system, the $\mathcal{H}_\infty$ synthesis problem can be stated as follows: Find an output feedback control matrix F that minimizes the $\mathcal{H}_\infty$ norm of a certain transfer function subject to the constraint that this control matrix F is stabilizing the associated control system. A typical instance of an output feedback control system can be stated as follows. Consider the LTI control system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + B_1 w(t), \\ y(t) &= Cx(t), \\ z(t) &= C_1 x(t) + D_1 u(t), \end{aligned} \tag{2.1}$$

where $x \in \mathbb{R}^{n_x}$, $w \in \mathbb{R}^{n_w}$, $u \in \mathbb{R}^{n_u}$, $z \in \mathbb{R}^{n_z}$, and $y \in \mathbb{R}^{n_y}$ denote the state, the disturbance input, the control input, the regulated output, and the measured output, respectively. Furthermore, A, B_1, B, C_1, C, and D_1 are given constant matrices of appropriate dimensions.

We consider the static output feedback (SOF) control law:

$$u(t) = Fy(t), \tag{2.2}$$

where $F \in \mathbb{R}^{n_u\times n_y}$ denotes the unknown static output feedback gain, which we attempt to compute by a suitable numerical procedure.

Given an output feedback matrix F and a control system (2.1), the closed-loop counterpart is given by:

$$\begin{aligned} \dot{x}(t) &= A(F)x(t) + B(F)w(t), \\ z(t) &= C(F)x(t), \end{aligned} \tag{2.3}$$

where

$$A(F) := A + BFC, \quad B(F) := B_1, \quad C(F) := C_1 + D_1 FC$$

are the augmented closed loop operators, respectively.

The following assumption is needed throughout the paper.