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A NEW ALGORITHM FOR COMPUTING THE INVERSE AND GENERALIZED INVERSE OF THE SCALED FACTOR CIRCULANT MATRIX*

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Abstract

A new algorithm for finding the inverse of a nonsingular scaled factor circulant matrix is presented by the Euclid's algorithm. Extension is made to compute the group inverse and the Moore-Penrose inverse of the singular scaled factor circulant matrix. Numerical examples are presented to demonstrate the implementation of the proposed algorithm.

Mathematics subject classification: 15A21, 65F15.
Key words: Scaled factor circulant matrix, Inverse, Group inverse, Moore-Penrose inverse.

1. Introduction

Circulant matrices, as an important class of special matrices, have a wide range of interesting applications [12–19]. They have in recent years been applied in many areas, see, e.g., [2, 3, 6, 10, 11, 15, 17]. Scaled circulant permutation matrices and the matrices that commute with them are natural extensions of this well-studied class, see, e.g., [1, 20–23]. In particular, it will be seen that r-circulant matrices [10, 11] are precisely those matrices commuting with the scaled circulant permutation matrix.

This paper presents an efficient algorithm to compute the inverse of a nonsingular scaled factor circulant matrix or to compute the group inverse and Moore-Penrose inverse of the circulant matrix when it is singular. The algorithm has small computational complexity. It is a notable character of the algorithm that the singularity of the scaled factor circulant matrix need not be priori known.

We define $\mathcal{R}$ as the scaled circulant permutation matrix, that is,

$$\mathcal{R} = \begin{pmatrix} 0 & d_1 & 0 & \dots & 0 & 0 \\ 0 & 0 & d_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & d_{n-1} \\ d_n & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{n\times n} . \tag{1.1}$$

This paper deals with the case where $\mathcal{R}$ is nonsingular ($d_i \neq 0$ and fixed).

It is easily verified that the polynomial $g(x) = x^n - d_1 d_2 \dots d_n$ is both the minimal polynomial and the characteristic polynomial of the matrix $\mathcal{R}$. In addition, $\mathcal{R}$ is nondergatory.

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Moreover, $\mathcal{R}$ is normal if and only if $|d_1| = |d_2| = \cdots = |d_n|$, where $|d_i|, i = 1, \cdots, n$ denote the modulus of the complex number $d_i, i = 1, \cdots, n$.

Definition 1.1. *An $n \times n$ matrix A over $\mathbb{C}$ is called a scaled factor circulant matrix if A commutes with $\mathcal{R}$, that is,*

$$A\mathcal{R} = \mathcal{R}A, \tag{1.2}$$

where $\mathcal{R}$ is given in (1.1).

Let $\mathcal{R}SFCM_n$ be the set of all complex $n \times n$ matrices which commute with $\mathcal{R}$. In the following, with $A = \text{scacirc}_{\mathcal{R}}(a_0, a_1, \cdots, a_{n-1})$ we denote the scaled factor circulant matrix A whose first row is $(a_0, a_1, \ldots, a_{n-1})$. Remark that the first row of A completely defines the matrix. Indeed, since $\mathcal{R}$ is nonderogatory, Eq. (1.2) is fulfilled if and only if $A = f(\mathcal{R})$ for some polynomial f. Furthermore, $\mathcal{R}SFCM_n$ is a vector space of dimension n, and there is a clear one-to-one correspondence between the polynomials of degree at most $n-1$ and the numbers $a_0, \cdots, a_{n-1}$.

For an $m \times n$ matrix A, any solution to the matrix equation $AXA = A$ is called a *generalized inverse* of A. In addition, if X satisfies $X = XAX$, then A and X are said to be semi-inverses, see, e.g., [2].

In this paper we only consider square matrices A. In [8, p.51] the smallest positive integer k for which rank(A^{k+1})=rank(A^k) holds is called the *index* of A. If A has index 1, the generalized inverse X of A is called the *group inverse* $A^{\#}$ of A. Clearly, A and X are group inverses if and only if they are semi-inverses and $AX = XA$.

In [4, 5] a semi-inverse X of A was considered in which the nonzero eigenvalues of X are the reciprocals of the nonzero eigenvalue of A. These matrices were called *spectral inverses*. It was shown in [5] that a nonzero matrix A has a unique spectral inverse, A^s, if and only if A has index 1: when A^s is the group inverse $A^{\#}$ of A.

2. The Properties of the Scaled Factor Circulant Matrix

Lemma 2.1. ([1]) *If $\mathcal{R}$ is a scaled circulant permutation matrix, and if k is a positive integer, then $\mathcal{R}^k = D^{(k)}C^k$, where $D^{(k)}$ is the diagonal matrix whose (j,j) entry is $\prod_{t=j}^{j+k-1} d_t$ for $1 \le j \le n$ and $C = circ(0, 1, 0, \cdots, 0)$ is the circulant permutation. Furthermore,*

$$\mathcal{R}^n = (\prod_{j=1}^{n} d_j) I_n, \quad \det \mathcal{R} = (-1)^{n-1} \prod_{j=1}^{n} d_j.$$

Let $\omega = \exp(\frac{2\pi i}{n})$ be a primitive nth root of unity. Then $\omega_j = d\omega^j, j = 0, 1, \cdots, n-1$ are the distinct roots of $g(x)$, where $g(x) = x^n - d_1 d_2 \cdots d_n$, and

$$d = (\prod_{t=1}^{n} d_t)^{\frac{1}{n}} \neq 0. \tag{2.1}$$

Let F be the $n \times n$ unitary Fourier matrix such that

$$F_{ij} = \frac{1}{\sqrt{n}}\, \omega^{(i-1)(j-1)} \quad \text{for } 1 \le i,\ j \le n. \tag{2.2}$$

Let

$$\Delta = \text{diag}(\delta_1, \delta_2, \cdots, \delta_n), \tag{2.3}$$