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A PERTURBATION METHOD FOR THE NUMERICAL SOLUTION OF THE BERNOULLI PROBLEM*

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Abstract

We consider the numerical solution of the free boundary Bernoulli problem by employing level set formulations. Using a perturbation technique, we derive a second order method that leads to a fast iteration solver. The iteration procedure is adapted in order to work in the case of topology changes. Various numerical experiments confirm the efficiency of the derived numerical method.

Mathematics subject classification: 35R35, 34E10, 65M06.
Key words: Bernoulli problem, Free boundary, Level sets.

1. Introduction

The Bernoulli problem stands for a prototype of a large class of stationary free boundary problems involved in fluid dynamics and electromagnetic shaping (see [5, 6, 8] and the references therein). This problem roughly consists in a Laplace equation with an additional boundary condition that enables determining the solution of the equation as well as the unknown domain.

In order to obtain a reliable numerical approximation for this problem, a wide variety of works have been produced. For instance, in Flucher and Rumpf [7], some numerical schemes based on a local parametrization are developed. The authors prove in this work convergence results and present some numerical examples. Nevertheless, due to the local parametrization, the constructed methods cannot handle topological changes. In [3], we propose an extension of the Flucher-Rumpf technique introducing a level set formulation to characterize the free boundary. This approach enjoys the property of allowing topology changes as level sets generally do. However, the scheme developed in [3] has the drawback to slowly converge and produces some local oscillation of the computed boundary when the numerical solution approaches the steady state. This drawback is removed in [11] where the authors consider an integral formulation of the Bernoulli problem and where the level set equation is solved via the Fast Marching strategy. The integral representation is however specific to partial differential equations for which this is available.

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In order to improve the solver performances, we propose in this paper a second-order scheme that can be viewed as a Newton-like method. The method is based on a perturbation of the parametrization of the initial guess of the free boundary. It has, as will be shown, the advantage of accelerating the convergence to the steady state solution, but as high-order methods require additional regularity properties, the presented method fails to converge when a topology change occurs during the iteration process. We then resort to switching to the first-order method while a domain splits up or two subdomains collapse. Numerical experiments show that convergence properties are dramatically improved when compared to the algorithm developed in [3].

The outline of the paper is as follows: In Section 2, we present a perturbation method to derive a second-order formulation. Section 3 is devoted to the derivation of a numerical scheme based on level sets and inspired by this perturbation technique. Section 4 presents some numerical results for both a radial case for which the analytical solution is known and a case with changing topology. Finally, a conclusion is drawn.

Let us mention that only the interior Bernoulli problem (see [3] for instance) is considered in the present study. An analog analysis of the exterior problem can be deduced straightforwardly.

2. The Perturbed Problem

Let Ω be a bounded domain of $\mathbb{R}^2$ with a C^2-boundary $\partial\Omega$. We seek a (not necessarily connected) domain A with $\bar{A} \subset \Omega$ and a function u defined on $\Omega \setminus A$ such that:

$$\Delta u = 0 \qquad \text{in } \Omega \setminus \bar{A}, \tag{2.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2.2}$$

$$u = 1 \qquad \text{on } \partial A, \tag{2.3}$$

$$\frac{\partial u}{\partial n} = \lambda \qquad \text{on } \partial A, \tag{2.4}$$

where λ is a positive real number and n is the unit normal to the boundary $\partial(\Omega \setminus \bar{A})$ of $\Omega \setminus \bar{A}$ pointing inward A.

We propose, in this section, an alternative to the result obtained in by Flucher and Rumpf (see [7], Theorem 2).

Proposition 2.1. *Let $\partial\tilde{A} = \partial A + \rho n$ be a set close to A (in the sense that $\rho \ll 1$). Then the function u — extended to $\Omega \setminus \tilde{A}$ if necessary — is solution to the following problem:*

$$\Delta u = 0 \qquad \text{in } \Omega \setminus \bar{\tilde{A}}, \tag{2.5}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{2.6}$$

$$\frac{\partial u}{\partial \tilde{n}} - \tilde{\kappa} u = \lambda - \tilde{\kappa} + \mathcal{O}(\rho^2) \qquad \text{on } \partial\tilde{A}, \tag{2.7}$$

where $\tilde{\kappa}$ is the curvature of $\partial\tilde{A}$.

To prove this result, we first need to consider some preliminary results.

2.1. Some technical results

Let $\gamma : [0, L] \to \mathbb{R}^2$ denote a parametric representation of the curve ∂A. We choose the parametrization such that the unit normal vector n to ∂A points inward A. The tangent vector