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ERROR ANALYSIS FOR A FAST NUMERICAL METHOD TO A BOUNDARY INTEGRAL EQUATION OF THE FIRST KIND*

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Abstract

For two-dimensional boundary integral equations of the first kind with logarithmic kernels, the use of the conventional boundary element methods gives linear systems with dense matrix. In a recent work [*J. Comput. Math.*, 22 (2004), pp. 287-298], it is demonstrated that the dense matrix can be replaced by a sparse one if appropriate graded meshes are used in the quadrature rules. The numerical experiments also indicate that the proposed numerical methods require less computational time than the conventional ones while the formal rate of convergence can be preserved. The purpose of this work is to establish a stability and convergence theory for this fast numerical method. The stability analysis depends on a decomposition of the coefficient matrix for the collocation equation. The formal orders of convergence observed in the numerical experiments are proved rigorously.

Mathematics subject classification: 65R20, 45L10.
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1. Introduction

Consider the first-kind boundary integral equation of the form

$$-\int_\Gamma \log|\mathbf{x}-\mathbf{y}|u(\mathbf{y})\, ds_{\mathbf{y}} = f(\mathbf{x}), \qquad \mathbf{x} := (x_1, x_2) \in \Gamma, \tag{1.1}$$

where $\Gamma \subset \mathbb{R}^2$ is a smooth and closed curve in the plane, u is a unknown function, f is a given function, $|\mathbf{x}-\mathbf{y}|$ denotes the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$, and $ds_{\mathbf{y}}$ is the measure of arclength. The boundary integral equation (1.1) arises in connection with the single layer potential method for

$$\Delta v(\mathbf{x}) = 0, \quad \mathbf{x} \in \Omega; \quad v(\mathbf{x}) = u(\mathbf{x}), \quad \mathbf{x} \in \Gamma, \tag{1.2}$$

whose solution can be represented by

$$v(\mathbf{x}) = -\int_\Gamma \log|\mathbf{x}-\mathbf{y}|u(\mathbf{y})\, ds_{\mathbf{y}}, \qquad \mathbf{x} \in \Omega. \tag{1.3}$$

Thus sampling of (1.3) on the boundary leads to the boundary integral equation (1.1). If the boundary Γ is sufficiently smooth, then the solution $v(\mathbf{x})$ can be very smooth due to the

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connection of the solutions of (1.2) and (1.3). The applications and some numerical aspects of the boundary integral equation (1.1) can be found in Sloan [14]. A more relevant paper by Bialecki and Yan [3] introduced a rectangular quadrature method for (1.1). More recently, Cheng et al. [6] proposed a new quadrature method for (1.1) based on a *graded mesh* approach. Unlike the quadrature method in [3] and other traditional numerical methods, the resulting system of equations in [6] contains a sparse coefficient matrix. It was demonstrated numerically that the proposed approach can not only preserve the formal rate of convergence but also save a significant amount of computational time.

The purpose of this paper is to provide a convergence theory for the method proposed in [6]. To begin with, let Γ be parameterized by the arclength:

$$\nu: \ [-L/2, L/2] \to \Gamma,$$

where L is the length of Γ,

$$|d\nu/ds| = 1 \text{ and } \nu(\sigma) \text{ is a periodic function with period of } L. \tag{1.4}$$

Then the integral equation (1.1) is equivalent to

$$-\int_{-L/2}^{L/2} \log|\nu(s) - \nu(\sigma)| u\big(\nu(\sigma)\big)\, d\sigma = f\big(\nu(s)\big), \qquad s \in [-L/2, L/2]. \tag{1.5}$$

The conventional way in solving Eq. (1.5) is to obtain n collocation equations by using n collocation points. Then for each fixed s the integral in (1.5) is approximated by an appropriate quadrature rule using the information on the n collocation points. This approach will lead to a linear system with a full matrix. In [6], the integral term in (1.5) is approximated by using a subset of the n collocation points. More precisely, let us consider the case when the unknown function u is reasonably smooth and the curve Γ is smooth and closed. In this case, some suitable graded-meshes can be used as the quadrature points to handle the logarithmic kernel, which yields a linear system with a sparse matrix. The graded-mesh concept was proposed by Rice [12]. It was then used to improve the formal order of convergence when solutions have weak singularity, see, e.g., [7, 19] for boundary integral equations and [4, 5, 15, 16] for weakly *singular* Volterra equations. However, with a smooth solution we just need to use a uniform mesh for the collocation points; while the graded mesh which is a subset of the uniform mesh is employed to evaluate the integrals.

To be more specific of numerical techniques, let us first introduce some notations. Set the *uniform mesh* with the mesh points

$$A := \{\alpha_i\}, \qquad \alpha_i = \frac{2i}{n-1} \cdot \frac{L}{2} \quad (i = -(n-1)/2, \cdots, (n-1)/2), \tag{1.6}$$

where n is supposed to be odd; and set the *graded mesh* with the mesh points

$$B := \{\beta_j\}, \qquad \beta_j = \operatorname{sgn}(j) \left(\frac{2|j|}{m}\right)^q \cdot \frac{L}{2} \quad (j = -m/2, \cdots, -1, 1, \cdots, m/2), \tag{1.7}$$

where $q \geq 1$ is the grading exponent. For ease of finding the mesh point set $B \subset A$, the value of q is usually taken as even. In this paper, we analyze the result for $q = 2$ and $q = 4$. For $q = 2$, it is assumed that $m = \sqrt{n-1}$. It can be verified that $B \subset A$. Transforming the negative index in (1.6) and (1.7) to positive one, we obtain the equivalent mesh-point sets:

$$\bar{A} := \{\bar{\alpha}_i\}, \qquad \bar{\alpha}_i = \alpha_{(i-1)-(n-1)/2} \quad (i = 1, \cdots, n), \tag{1.8}$$