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A MONOTONE DOMAIN DECOMPOSITION ALGORITHM FOR SOLVING WEIGHTED AVERAGE APPROXIMATIONS TO NONLINEAR SINGULARLY PERTURBED PARABOLIC PROBLEMS*

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Abstract

This paper presents and analyzes a monotone domain decomposition algorithm for solving nonlinear singularly perturbed reaction-diffusion problems of parabolic type. To solve the nonlinear weighted average finite difference scheme for the partial differential equation, we construct a monotone domain decomposition algorithm based on a Schwarz alternating method and a box-domain decomposition. This algorithm needs only to solve linear discrete systems at each iterative step and converges monotonically to the exact solution of the nonlinear discrete problem. The rate of convergence of the monotone domain decomposition algorithm is estimated. Numerical experiments are presented.

Mathematics subject classification: 65M06, 65M12, 65M55.
Key words: Parabolic reaction-diffusion problem, Boundary layers, θ-method, Monotone domain decomposition algorithm, Uniform convergence.

1. Introduction

We are interested in monotone iterative methods for solving nonlinear singularly perturbed problems which correspond to a reaction-diffusion problem of parabolic type

$$-\mu^2(u_{xx}+u_{yy})+u_t=-f(x,y,t,u), \tag{1.1}$$
$$(x,y,t)\in Q=\omega\times(0,T], \quad \omega=\{0<x<1\}\times\{0<y<1\},$$
$$0\le f_u\le c^*=\text{const}, \quad (x,y,t,u)\in\overline{Q}\times(-\infty,\infty), \quad (f_u\equiv\partial f/\partial u), \tag{1.2}$$

where μ is a small positive parameter. The initial-boundary conditions are

$$u(x,y,0)=u^0(x,y), \quad (x,y)\in\overline{\omega},$$
$$u=g, \quad (x,y,t)\in\partial\omega\times(0,T],$$

where $\partial\omega$ is the boundary of $\overline{\omega}$. If f, g and u^0 are sufficiently smooth, then under suitable continuity and compatibility conditions on the data, a unique solution u of (1.1) exists [6]. We mention that the assumption $f_u\ge 0$ in (1.1) can always be obtained via a change of variables.

For $\mu\ll 1$, the reaction-diffusion problem (1.1) is singularly perturbed and characterized by the boundary layers of width $\mathcal{O}(\mu|\ln\mu|)$ near $\partial\omega$, see, e.g., [1].

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We shall employ the weighted average scheme for solving problem (1.1). This nonlinear ten-point difference scheme can be regarded as taking a weighted average of the explicit and implicit schemes. It was proved in [2] that, for certain piecewise equidistant spatial meshes, the weighted average scheme converges μ-uniformly to the exact solution of problem (1.1). The truncation error analysis of [4] proved μ-uniform convergence on special fitted piecewise uniform and log-meshes.

In order to practically compute the nonlinear weighted average scheme, one requires a robust and efficient algorithm. A fruitful method is the method of upper and lower solutions and its associated monotone iterations. Since the initial data in the monotone iterative method is either an upper or lower solution, constructed directly from the difference equations without any knowledge of the exact solution, this method simplifies the search for the initial data and thus gives a practical advantage over Newton's method in the computation of numerical solutions. Based on the method of upper and lower solutions, the monotone iterative method of [2] converges μ-uniformly to the solution of problem (1.1) and only requires the solution of linear systems at each iterative step. The numerical experiments of [2] and [4] confirmed the theoretical rates of convergence on piecewise equidistant and log-meshes.

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures have received much attention for their potential as efficient algorithms for parallel computing. In [3], we proposed an iterative algorithm for solving the nonlinear implicit finite difference scheme approximation of the partial differential equation (1.1). This algorithm combines the monotone approach and an iterative domain decomposition method based on the Schwarz alternating procedure. The spatial computational domain is partitioned into nonoverlapping box-subdomains. At each horizontal and vertical boundary, a small interfacial subdomain is introduced and an associated linear problem generates boundary values for the nonoverlapping box-subdomains. Thus, this approach may be considered as a variant of a block Gauss-Seidel iteration (or in the parallel context as a multicoloured algorithm) for the nonoverlapping box-subdomains with a Dirichlet-Dirichlet coupling through the interface variables. In this paper, we generalize the monotone box-domain decomposition algorithm of [3] from the nonlinear implicit scheme to the nonlinear weighted average scheme.

The structure of the paper is as follows. In Section 2, we present the nonlinear weighted average scheme and discuss the stability of two different weightings. Section 3 proposes a monotone domain decomposition algorithm based on the box-domain decomposition from [3]. We develop estimates of the rate of convergence and prove that on the piecewise uniform meshes the monotone domain decomposition algorithm converges μ-uniformly to the solution of (1.1). The numerical experiments of Section 4 correlate the convergence behaviour of the algorithm with the theoretical convergence parameter derived in Section 3. The experiments demonstrate the surprising result that, for sufficiently small perturbation parameter μ, this paper's domain decomposition generalization of the algorithm from [2] executes more quickly than the original undecomposed algorithm.

2. A Weighted Average Scheme

On $\overline{Q}$ introduce a rectangular mesh $\overline{\omega}^h \times \overline{\omega}^\tau$, $\overline{\omega}^h = \overline{\omega}^{hx} \times \overline{\omega}^{hy}$:

$$\overline{\omega}^{hx} = \{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\}, \tag{2.1a}$$

$$\overline{\omega}^{hy} = \left\{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\right\}, \tag{2.1b}$$

$$\overline{\omega}^{\tau} = \{t_k = k\tau,\ 0 \le k \le N_\tau,\ N_\tau \tau = T\}. \tag{2.1c}$$