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COMPACT FOURTH-ORDER FINITE DIFFERENCE SCHEMES FOR HELMHOLTZ EQUATION WITH HIGH WAVE NUMBERS*

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Abstract

In this paper, two fourth-order accurate compact difference schemes are presented for solving the Helmholtz equation in two space dimensions when the corresponding wave numbers are large. The main idea is to derive and to study a fourth-order accurate compact difference scheme whose leading truncation term, namely, the $\mathcal{O}(h^4)$ term, is independent of the wave number and the solution of the Helmholtz equation. The convergence property of the compact schemes are analyzed and the implementation of solving the resulting linear algebraic system based on a FFT approach is considered. Numerical results are presented, which support our theoretical predictions.

Mathematics subject classification: 65M06, 65N12.
Key words: Helmholtz equation, Compact difference scheme, FFT algorithm, Convergence.

1. Introduction

In this paper, we consider two-dimensional Helmholtz equation

$$\nabla^2 u + k^2 u = f(x, y), \tag{1.1}$$

where k is wave number, together with some appropriate boundary conditions. Boundary value problems governed by the Helmholtz equation describe many physical phenomena and have important applications in acoustic and electromagnetic waves.

When the wave number k is very large, Eq. (1.1) has a great difficulty in computation because in this case the solutions of Eq. (1.1) are highly oscillatory. There exist many different numerical methods for solving the Helmholtz equation, such as Galerkin finite element method [1], spectral method [8,12] and finite difference method [2,4,14]. Many of the proposed schemes can provide very accurate approximations to the highly oscillatory solutions under the condition that kh is very small, where h is a characteristic spatial grid size. This condition shows that in order to attain accurate approximate solutions, it is required to significantly decrease h with large wave number k.

On the other hand, in recent years, high-order accurate compact finite difference methods have been used widely for solving convection-diffusion problems, the Navier-Stokes equations and the Helmholtz equation [3-7,10,11,13,14]. This class of methods is attractive since they offer a means to obtain high accuracy solutions with less computational costs. In this paper, we will use the compact finite difference methods to deal with Eq. (1.1). We first derive two fourth-order compact finite difference schemes for the problem (1.1), and then provide some convergence analysis for the two methods. The main difference of the two proposed schemes is

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about the coefficient of the leading truncation errors: the coefficient of one of the schemes is independent of the wave number k and the solution of (1.1) (the solution is in general depends on k also). Consequently, it is expected that this scheme will be useful for solving Eq. (1.1) with large wave number k. Moreover, in this work we also apply the fast Fourier transform (FFT) algorithm to solve the algebraic system resulting from the compact finite difference discretizations. This significantly speeds up the computational efficiency.

The rest of the paper is organized as follows. In Section 2, two fourth-order compact finite difference schemes are presented. In Section 3, the convergence analysis of the proposed schemes for one- and two-dimensional Helmholtz equation is provided. Numerical implementation based on a FFT approach is given in Section 4. In Section 5, numerical experiments are carried out to verify the theoretical predictions obtained in this work.

2. Fourth-order Compact Schemes

We consider Eq. (1.1) with Dirichlet and Neumann boundary conditions. For ease of notations, we only consider a simple square domain $\Omega = (0,1)\times(0,1)$ with $\Delta x = \Delta y$, but the main ideas in this work can be extended to rectangular domains with $\Delta x \neq \Delta y$. Divide uniformly Ω with lines $\{(x_i, y_j) : x_i = ih, y_j = jh, i, j = 0, 1, \cdots, J\}$, where h is the spacial mesh-size. Use the notation δ_x^2, δ_y^2 to denote the second-order central difference with respect to x, y, respectively:

$$\delta_x^2 u_{i,j} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}, \quad \delta_y^2 u_{i,j} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{h^2}.$$

By the Taylor series expansion, we get for every sufficiently smooth u

$$\delta_x^2 u = u_{xx} + \frac{h^2}{12} u_{x^4} + \frac{h^4}{360} u_{x^6} + \mathcal{O}(h^6),$$

here and below for simplicity, we omit subscripts i, j whenever confusions will not occur. Adding a similar expression for δ_y^2 and rearranging the resulting terms give

$$u_{xx} + u_{yy} = (\delta_x^2 + \delta_y^2)u - \frac{h^2}{12}(u_{x^4} + u_{y^4}) - \frac{h^4}{360}(u_{x^6} + u_{y^6}) + \mathcal{O}(h^6). \tag{2.1}$$

Similarly, there is the expression

$$u_{x^2y^2} = \delta_x^2 \delta_y^2 u - \frac{h^2}{12}(u_{x^4y^2} + u_{x^2y^4}) + \mathcal{O}(h^4). \tag{2.2}$$

Inserting (2.1) into the Helmholtz equation (1.1) gives

$$(\delta_x^2 + \delta_y^2)u - \frac{h^2}{12}(u_{x^4} + u_{y^4}) - \frac{h^4}{360}(u_{x^6} + u_{y^6}) + k^2 u = f + \mathcal{O}(h^6). \tag{2.3}$$

In order to obtain the fourth-order accuracy, we need to approximate the term $u_{x^4} + u_{y^4}$ to $\mathcal{O}(h^2)$. By using the original equation (1.1) and the expressions (2.1), (2.2), we have

$$\begin{aligned} u_{x^4} + u_{y^4} &= \Delta^2 u - 2u_{x^2y^2} = \Delta(f - k^2 u) - 2u_{x^2y^2} = \Delta f - k^2(\delta_x^2 + \delta_y^2)u \\ &\quad -2\delta_x^2\delta_y^2 u + \frac{k^2h^2}{12}(u_{x^4} + u_{y^4}) + \frac{h^2}{6}(u_{x^2y^4} + u_{y^2x^4}) + \mathcal{O}(h^4). \end{aligned} \tag{2.4}$$