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AN IMPROVED ERROR ANALYSIS FOR FINITE ELEMENT APPROXIMATION OF BIOLUMINESCENCE TOMOGRAPHY*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

This paper is concerned with an ill-posed problem which results from the area of molecular imaging and is known as BLT problem. Using Tikhonov regularization technique, a quadratic optimization problem can be formulated. We provide an improved error estimate for the finite element approximation of the regularized optimization problem. Some numerical examples are presented to demonstrate our theoretical results.

Mathematics subject classification: 49J20, 65N30.
Key words: BLT problem, Tikhonov regularization, Optimization problem, A priori error estimate.

1. Introduction

In modern medical science, molecular imaging plays an important role. The traditional imaging techniques such as computed tomography (CT), magnetic resonance imaging (MRI) [12] can not fulfill the requirements, so optical imaging methods such as florescence molecular tomography (FMT) [13] and bioluminescence imaging (BLT) [14] are becoming flourishing in the decades. Bioluminescence imaging is based on the use of a family of enzymes known as luciferases, which are found in organisms that emit a bioluminescent glow. It can be applied to all disease processes in all areas of small-animal models. Examples of ongoing applications include cancer, inflammatory disease, neurodegenerative disease, gastrointestinal physiology, renal physiology, cell trafficking, stem cell research, transplant science, and muscle physiology.

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The bioluminescent photon transport in the media can be described by radiative transfer equation, which can be reduced to a diffusion equation [12]:

$$-\text{div}(D\nabla y) + \mu y = u \quad \text{in} \;\; \Omega, \tag{1.1}$$

$$y + 2(D\nabla y)\cdot n = g^- \quad \text{on} \;\; \partial\Omega. \tag{1.2}$$

We should find a source function u in view of the boundary value problems (1.1)-(1.2) and accordant with the measurement g on the boundary by $(D\nabla y)\cdot n = -g$. This is a typical inverse source problem and is proved strongly ill-posed. Through a Tikhonov regularization technique, a well-posed optimization problem to approximate the original BLT problem is proposed in [5], both convergence analysis and numerical treatment are provided.

In this paper, we analyze the regularized optimization scheme proposed in [5] using the analysis technique for optimal control problems. By introducing an adjoint state, the optimization problem can be converted to a system with three coupled equations. Then the analysis for both the continuous and discretized systems are clearer and easier. Using the a new methodology in the a priori error analysis for the finite element approximation of the regularized optimization problem, an improved error estimate is obtained comparing to the results in [5]. Numerical experiments confirm our results.

The paper is organized as follows: In Section 2, we introduce the mathematical model of bioluminescence tomography. In Section 3, the finite element scheme of the model problem is presented. Then the improved a priori error analysis is provided in Section 4. In the last section, some numerical results on the model problem are provided.

2. The Mathematical Model of Bioluminescence Tomography

Let us consider the following ill-posed problem:

$$-\text{div}(D\nabla y) + \mu y = Bu \quad \text{in} \;\; \Omega, \tag{2.1}$$

$$y + 2(D\nabla y)\cdot n = g^- \quad \text{on} \;\; \partial\Omega, \tag{2.2}$$

$$(D\nabla y)\cdot n = -g \quad \text{on} \; \partial\Omega, \tag{2.3}$$

where Ω is a bounded domain in $\mathbb{R}^n (n \le 3)$ with a Lipschitz boundary $\Gamma = \partial\Omega$, D is a symmetric or nonsymmetric positive definite matrix, $\mu \ge 0$, n is the outward normal on $\partial\Omega$, B is a linear operator from Ω_U to Ω, which has the typical form of a characteristic function χ_{Ω_U} on $\Omega_U \subset \Omega$.

In above problem, g^- is usually a given function and is zero in a typical BLT problem, whereas g is the measurement. We should detect the source function u by the measurement g, and u is usually in a closed convex subset Q_U of the space $L^2(\Omega_U)$. In the typical BLT problems Q_U has usually the form of $L^2(\Omega_U)$ or the subset of $L^2(\Omega_U)$ with nonnegatively valued functions. From the boundary conditions (2.2) and (2.3) we can formulate another boundary condition:

$$y = g^- + 2g \quad \text{on} \;\; \Gamma.$$

Then we will only consider the following two boundary conditions to fix the idea

$$y = g_1 \quad \text{on} \;\; \Gamma, \tag{2.4}$$

$$(D\nabla y)\cdot n = g_2 \quad \text{on} \;\; \Gamma. \tag{2.5}$$