Journal of Computational Mathematics, Vol.26, No.3, 2008, 437–455.

UNIFORMLY-STABLE FINITE ELEMENT METHODS FOR DARCY-STOKES-BRINKMAN MODELS*

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Dedicated to Professor Junzhi Cui on the occasion of his 70th birthday

Abstract

In this paper, we consider 2D and 3D Darcy-Stokes interface problems. These equations are related to Brinkman model that treats both Darcy's law and Stokes equations in a single form of PDE but with strongly discontinuous viscosity coefficient and zeroth-order term coefficient. We present three different methods to construct uniformly stable finite element approximations. The first two methods are based on the original weak formulations of Darcy-Stokes-Brinkman equations. In the first method we consider the existing Stokes elements. We show that a stable Stokes element is also uniformly stable with respect to the coefficients and the jumps of Darcy-Stokes-Brinkman equations if and only if the discretely divergence-free velocity implies almost everywhere divergence-free one. In the second method we construct uniformly stable elements by modifying some well-known $H(\mathbf{div})$-conforming elements. We give some new 2D and 3D elements in a unified way. In the last method we modify the original weak formulation of Darcy-Stokes-Brinkman equations with a stabilization term. We show that all traditional stable Stokes elements are uniformly stable with respect to the coefficients and their jumps under this new formulation.

Mathematics subject classification: 65N12, 65N15, 65N22, 65N30.
Key words: Darcy-Stokes equation, Brinkman, Finite element, Uniformly stable.

1. Introduction

In this paper, we consider the following model equations on a bounded, connected, and polygonal domain $\Omega \subset R^d$ $(d = 2, 3)$ (Fig. 1.1 is an example of two dimensional domain). A velocity $\mathbf{u}$ and a pressure p satisfy

$$\begin{cases} -\nabla \cdot (\nu(x)\nabla \mathbf{u}) + \alpha(x)\mathbf{u} + \nabla p = \mathbf{f} & \text{in} \quad \Omega, \\ \nabla \cdot \mathbf{u} = g & \text{in} \quad \Omega, \\ \mathbf{u} = \mathbf{0} & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

with piecewise-constant viscosity coefficient

$$\nu(x) = \nu_i > 0, \qquad x \in \Omega_i, \tag{1.2}$$

* Received March 15, 2008 / accepted March 29, 2008 /

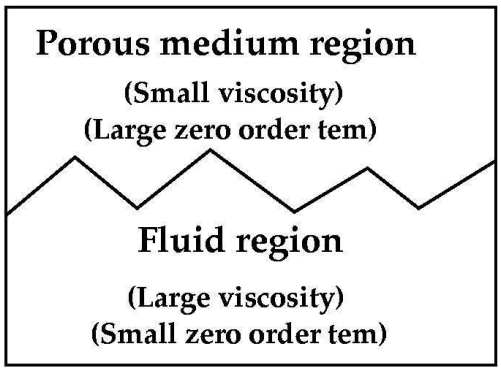

Fig. 1.1. Domain.

and piecewise-constant zeroth-order term coefficient

$$\alpha(x) = \alpha_i \geq 0, \qquad x \in \Omega_i. \tag{1.3}$$

The sub-domains Ω_i are assumed to be bounded connected polygonal domains such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $\overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega}_i$. By Γ_{ij}, we denote the interface between two adjacent sub-domains Ω_i and Ω_j, namely, $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. For other notations: $\sigma(\mathbf{u}, p) = \nu(x)\nabla\mathbf{u} - p\mathbf{I}$ is a stress tensor; $\mathbf{n}$ is the unit normal vector to $\Gamma_{i,j}$;

$$[\mathbf{u}]\,|_{\Gamma_{ij}} = \mathbf{u}|_{\partial\Omega_i \cap \Gamma_{ij}} - \mathbf{u}|_{\partial\Omega_j \cap \Gamma_{ij}}; \quad [\sigma(\mathbf{u}, p)\mathbf{n}]\,|_{\Gamma_{ij}} = \sigma(\mathbf{u}, p)\mathbf{n}|_{\partial\Omega_i \cap \Gamma_{ij}} - \sigma(\mathbf{u}, p)\mathbf{n}|_{\partial\Omega_j \cap \Gamma_{ij}}.$$

For the interface boundary conditions, we have $[\sigma(\mathbf{u}, p)\mathbf{n}]\,|_{\Gamma_{ij}} = 0$, and $[\mathbf{u}]\,|_{\Gamma_{ij}} = 0$. In addition, the source term g is assumed to satisfy the solvability condition:

$$\int_{\Omega} g dx = 0. \tag{1.4}$$

When α_i is big and ν_i is small in some sub-domains, the equation is close to Darcy equation; in some sub-domain where ν_i is big and α_i is small together with $g = 0$, the equation is close to the Stokes equation. This Darcy-Stokes equation is called Brinkman equation [1], which models porous media flow coupled with viscous fluid flow in a single form of equation.

Among many applications to the Darcy-Stokes-Brinkman equations, our motivation comes from computational fuel cell dynamics [2–4]. A fuel cell is a clean chemical energy conversion device which has potential to replace the traditional combustion engine. In the fuel cell, there are porous gas diffusion layers and gas channels. The two-phase mixture flow in the porous media is modeled by Darcy's law and flow in the gas channel is modeled by Navier-Stokes equations [5–10]. Reviews for this area can be found in [11, 12].

It is so-called single-domain approach that models multi-domain problems using single set of equations with highly discontinuous coefficients $\nu(x)$ and $\alpha(x)$. In this approach, the internal interface conditions are straightforward (the velocity and normal component of stress tensor are continuous), compared to other types of multi-domain Darcy-Stokes models that couple through three interface conditions [13–21].

The goal of this paper is to explore finite element methods which behave uniformly with respect to the highly discontinuous coefficients, $\nu(x)$ and $\alpha(x)$, and their jumps. We present three different methods.

In the first two methods, we consider the original weak formulation. As discussed above, our model problems can be reduced to two extreme cases. One is standard Stokes equation.