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A PRIORI ERROR ESTIMATE AND SUPERCONVERGENCE ANALYSIS FOR AN OPTIMAL CONTROL PROBLEM OF BILINEAR TYPE*

Danping Yang
Department of Mathematics, East China Normal University, Shanghai 200062, China
Email: dpyang@math.ecnu.edu.cn
Yanzhen Chang
School of Mathematics and System Science, Shandong University, Jinan 250100, China
Email: changyanzhen@mail.sdu.edu.cn
Wenbin Liu
KBS, University of Kent, Canterbury, CT2 7PE, England
Email: W.B.Liu@kent.ac.uk

Abstract

In this paper, we investigate a priori error estimates and superconvergence properties for a model optimal control problem of bilinear type, which includes some parameter estimation application. The state and co-state are discretized by piecewise linear functions and control is approximated by piecewise constant functions. We derive a priori error estimates and superconvergence analysis for both the control and the state approximations. We also give the optimal L^2-norm error estimates and the almost optimal L^∞-norm estimates about the state and co-state. The results can be readily used for constructing a posteriori error estimators in adaptive finite element approximation of such optimal control problems.

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1. Introduction

The finite element approximation of optimal control problems has been extensively studied in the literature. There have been extensive studies in convergence of the standard finite element approximation of optimal control problems, see, some examples in [2, 3, 9, 10, 18, 20, 23], although it is impossible to give even a very brief review here. For optimal control problems governed by linear state equations, a priori error estimates of the finite element approximation were established long ago; see, e.g., [9, 10]. But it is more difficult to obtain such error estimates for nonlinear control problems. For some classes of nonlinear optimal control problems, a priori error estimates were established in [4, 11, 17]. The optimal control problem of bilinear type considered in this paper includes a useful model problem of parameter estimation, and there does not seem to exist systematical studies in the literature on its finite element approximation and error analysis, except [14] where a posteriori error estimates were presented.

Furthermore superconvergence analysis is an important topic for finite element approximation of PDEs. Due to the lower regularity of the constrained optimal control (normally only $H^1(\Omega) \cap W^{1,\infty}(\Omega)$), only a half order of convergence rate can be expected to gain by using

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the standard recovery techniques, see [26]. Very recently Meyer and Rösch in [19] showed that in fact one order can be gained via using a special projection, which was unique to the linear optimal control problem they studied. This is a quite interesting result considering the low regularity of the optimal control. It is useful to establish such a superconvergence property for our model control problem, which is normally difficult to compute with higher accuracy.

In this paper we firstly study a priori error estimates, superconvergence analysis of the control problem with a projection and interpolator different from [19].

The plan of the paper is as follows. In Sections 2-3, we describe the control problem and give the finite element approximation. In Section 4, we derive a priori error estimates. In Section 5, superconvergence analysis is carried out. Then some applications and super-convergent result are discussed in Section 6.

2. Optimal Control Problem

In this section, we formulate the bilinear optimal control problem. Let Ω be a bounded convex polygon in R^2 with boundary $\partial\Omega$. We adopt the standard notation $W^{m,q}(\Omega)$ for Sobolev spaces on Ω with the norm $\|\cdot\|_{m,q,\Omega}$ and the seminorm $|\cdot|_{m,q,\Omega}$. We set $W_0^{m,q}(\Omega) \equiv \{w \in W^{m,q}(\Omega) : w|_{\partial\Omega} = 0\}$ and denote $W^{m,2}(\Omega)$ $(W_0^{m,2}(\Omega))$ by $H^m(\Omega)$ $(H_0^m(\Omega))$ with the norm $\|\cdot\|_{m,\Omega}$ and the seminorm $|\cdot|_{m,\Omega}$. We shall take the state space $V = H_0^1(\Omega)$, the control space $U = L^2(\Omega)$, and the observation space $Y = L^2(\Omega)$. Define the control constraint set $K \subset U$:

$$K = \{v \in U : \quad v \geq 0\}.$$

We are interested in the following optimal control problem of bilinear type :

$$\begin{aligned} &(a) \quad \min_{v\in K}\{\frac{1}{2}\|y - y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|v\|_{0,\Omega}^2\}, \\ &(b) \quad -\operatorname{div}(A\nabla y) + vy = f \text{ in } \Omega \ , \quad y|_{\partial\Omega} = 0, \end{aligned} \tag{2.1}$$

where α is a positive constant. $f \in L^2(\Omega)$ and $A(\cdot) = (a_{ij}(\cdot))_{2\times 2} \in [W^{1,\infty}(\Omega)]^{2\times 2}$ is a symmetric positive definite matrix. This problem can be interpreted as a model of estimating the true parameter u via the measured data y_d using the least square formulation.

To consider the finite element approximation of the above optimal control problem, we have to give a weak formula for the state equation. Let

$$\begin{aligned} a(y, w) &= \int_\Omega (A\nabla y)\cdot\nabla w \quad \forall\ y, w \in V, \\ (v, w) &= \int_\Omega vw \quad \forall\ v, w \in U. \end{aligned}$$

We assume that there are constants $a_0 > 0$ and $C_0 > 0$ such that

$$a_0\|y\|_V^2 \leq a(y, y), \quad |a(y, w)| \leq C_0\|y\|_V\|w\|_V, \quad \forall\ y, w \in V. \tag{2.2}$$

Then the standard weak formula for the state equation reads as follows: find $y(v) \in V$ such that

$$a(y(v), w) + (vy(v), w) = (f, w) \qquad \forall\ w \in V. \tag{2.3}$$

Introduce a cost function

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_{0,\Omega}^2 + \frac{\alpha}{2}\|v\|_{0,\Omega}^2. \tag{2.4}$$