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ON NON-ISOTROPIC JACOBI PSEUDOSPECTRAL METHOD*

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Abstract

In this paper, a non-isotropic Jacobi pseudospectral method is proposed and its applications are considered. Some results on the multi-dimensional Jacobi-Gauss type interpolation and the related Bernstein-Jackson type inequalities are established, which play an important role in pseudospectral method. The pseudospectral method is applied to a two-dimensional singular problem and a problem on axisymmetric domain. The convergence of proposed schemes is established. Numerical results demonstrate the efficiency of the proposed method.

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Key words: Jacobi pseudospectral method in multiple dimensions, Jacobi-Gauss type interpolation, Bernstein-Jackson type inequalities, Singular problem, Problem on axisymmetric domain.

1. Introduction

The main advantage of spectral method is its high accuracy, see [4–9]. However, this merit may be seriously affected by singularities of genuine solutions, which could be caused by several factors, such as degenerating coefficients of leading terms in differential equations. Moreover, the coefficients of derivatives of different orders involved in underlying problems may degenerate in completely different way. For solving such problems, Guo [11, 12] developed the Jacobi approximation in certain non-uniformly weighted Sobolev space, and proposed the corresponding Jacobi spectral method with its applications to one-dimensional singular differential equations. We also refer to the work on the Jacobi approximation in [1, 7, 16, 20]. The Jacobi spectral method is also very useful for many kinds of other related problems, e.g., differential equations on unbounded domains and axisymmetric domains, [3, 10, 13, 14, 25]. On the other hand, some results on the Jacobi approximation have been successfully applied to the analysis of various rational spectral methods,see e.g., [15, 17, 18, 22, 23, 26].

In practice, it is more important and interesting to solve multi-dimensional singular problems and related problems numerically. Guo and Wang [21] provided the Jacobi spectral method in two-dimensions. It is well known that the pseudospectral method is more preferable in actual computations, since it only needs to evaluate unknown functions at interpolation nodes. This feature simplifies calculations and saves a lot of work. Furthermore, it is much easier to deal with nonlinear terms. Guo and Wang [19] investigated the Jacobi pseudospectral method for one-dimensional singular problems. However, no existing works have been found for considering the Jacobi pseudospectral method in multiple dimensions.

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This paper is devoted to the Jacobi pseudospectral method in multiple dimensions and its applications. In the next section, we recall some basic results on the one-dimensional Jacobi approximation. In Section 3, we establish the main results on the Jacobi-Gauss type interpolation in multi-dimensional space, which play important role in designing and analyzing various Jacobi pseudospectral schemes for singular problems and other related problems. We also derive a series of sharp results on the Legendre-Gauss type interpolation and the related Bernstein-Jackson type inequalities, which are very useful for pseudospectral methods of partial differential equations with non-constant coefficients. It is noted that Canuto, Hussaini et.al [6], and Bernardi and Maday [4] first studied the Legendre-Gauss type interpolation in the Sobolev spaces, and Quarteroni [24] first considered the Bernstein-Jackson type inequalities in L_p-space for the Legendre orthogonal approximation. We improve and generalize some of those results in this paper. As examples of applications, we consider a two-dimensional singular problem in Section 4, and a problem defined on an axisymmetric domain in Section 5. The convergence of proposed schemes is proved. Numerical results confirm the theoretical predictions. The final section provides some concluding remarks.

2. Preliminaries

We first recall some basic results on the one-dimensional Jacobi approximation. Let $\Lambda = (-1, 1)$, and $\chi(x)$ be a certain weight function. Denote by $\mathbb{N}$ the set of all non-negative integers. For any $r \in \mathbb{N}$, we define the weighted Sobolev space $H^r_\chi(\Lambda)$ in the usual way, with the inner product $(u, v)_{r,\chi,\Lambda}$, semi-norm $|v|_{r,\chi,\Lambda}$ and norm $\|v\|_{r,\chi,\Lambda}$, respectively. In particular,

$$L^2_\chi(\Lambda) = H^0_\chi(\Lambda), \quad (u, v)_{\chi,\Lambda} = (u, v)_{0,\chi,\Lambda},$$

and $\|v\|_{\chi,\Lambda} = \|v\|_{0,\chi,\Lambda}$. For any $r > 0$, the space $H^r_\chi(\Lambda)$ and its norms are defined by space interpolation as in [2]. The space $H^r_{0,\chi}(\Lambda)$ stands for the closure in $H^r_\chi(\Lambda)$ of the set $\mathcal{D}(\Lambda)$ consisting of all infinitely differentiable functions with compact support in Λ. Besides,

$${}_0H^r_\chi(\Lambda) = \{v \mid v \in H^r_\chi(\Lambda), v(-1) = 0\}.$$

Whenever $\chi(x) \equiv 1$, we omit the subscript χ in the notations.

Let α, $\beta > -1$. The Jacobi polynomials $J^{(\alpha,\beta)}_l(x)$, $l = 0, 1, 2, \ldots$, are the eigenfunctions of Sturm-Liouville problem

$$\partial_x((1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x v(x)) + \lambda(1-x)^\alpha(1+x)^\beta v(x) = 0, \qquad x \in \Lambda,$$

with the corresponding eigenvalues

$$\lambda^{(\alpha,\beta)}_l = l(l + \alpha + \beta + 1) \quad l = 0, 1, 2, \ldots$$

Let $\chi^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$. We have that

$$\int_\Lambda J^{(\alpha,\beta)}_l(x) J^{(\alpha,\beta)}_{l'}(x) \chi^{(\alpha,\beta)}(x)\,\mathrm{d}x = \gamma^{(\alpha,\beta)}_l \delta_{l,l'},$$

where $\delta_{l,l'}$ is the Kronecker symbol, and

$$\gamma^{(\alpha,\beta)}_l = \frac{2^{\alpha+\beta+1}\Gamma(l+\alpha+1)\Gamma(l+\beta+1)}{(2l+\alpha+\beta+1)\Gamma(l+1)\Gamma(l+\alpha+\beta+1)}.$$