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ENERGY ESTIMATES FOR DELAY DIFFUSION-REACTION EQUATIONS*

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Abstract

In this paper we consider nonlinear delay diffusion-reaction equations with initial and Dirichlet boundary conditions. The behaviour and the stability of the solution of such initial boundary value problems (IBVPs) are studied using the energy method. Simple numerical methods are considered for the computation of numerical approximations to the solution of the nonlinear IBVPs. Using the discrete energy method we study the stability and convergence of the numerical approximations. Numerical experiments are carried out to illustrate our theoretical results.

Mathematics subject classification: 65M06, 65M20, 65M15.
Key words: Delay diffusion reaction equation, Energy method, Stability, Convergence.

1. Introduction

Initial boundary value problems with memory defined by the nonlinear delay diffusion-reaction equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t), u(x,t-\tau)), \qquad (x,t) \in (a,b) \times (0,T], \tag{1.1}$$

where $\tau > 0$ is a delay parameter, $\alpha > 0$, and by the conditions

$$u(a,t) = u_a(t),\ u(b,t) = u_b(t),\ t \in (0,T], \tag{1.2}$$

$$u(x,t) = u_0(x,t),\ x \in (a,b),\ t \in [-\tau, 0], \tag{1.3}$$

or systems of delay diffusion-reaction equations of type (1.1), are largely used on the description of biological phenomena. The simplest model is the one obtained replacing the diffusion Verhulst equation by the logistic delay equation (1.1) with the reaction term

$$f(u(x,t), u(x,t-\tau)) = ru(x,t)\big(1 - \frac{u(x,t-\tau)}{\beta}\big),$$

where r and β are positive constants. Other versions of Eq. (1.1) are considered to model growth population phenomena. For instance, the x-independent version of Eq. (1.1) with

$$f(u(x,t), u(x,t-\tau)) = be^{-au(x,t-\tau)-d_1\tau} u(x,t-\tau) - du(x,t),$$

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where a, b, d and d_1 are positive parameters, is proposed in [5] to study a grow birth population. Eq. (1.1) with

$$f(u(x,t), u(x,t-\tau)) = bu(x,t-\tau)(1-u(x,t)) - cu(x,t),$$

where b and c are positive parameters, is considered in [11] independent of x, to model epidemic propagations phenomena.

Systems of delay partial differential equations of type (1.1) have been also used to describe mathematically biological phenomena. In [13], the x-independent version of the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \alpha_1 \dfrac{\partial^2 u_1}{\partial x^2} - R_0 u_1(x,t) u_2(x,t-\tau) + u_2(x,t), \\ \dfrac{\partial u_2}{\partial t} = \alpha_2 \dfrac{\partial^2 u_2}{\partial x^2} + R_0 u_1(x,t) u_2(x,t-\tau) - u_2(x,t), \end{cases} \tag{1.4}$$

where u_1 and u_2 represent the ratio of susceptible and infected individuals and $\alpha_i, i=1,2$, R_0 are positive constants, was used to study an epidemic propagation.

All the models presented before have been based on the Fick's law for the flux combined with a mass conservation law. The memory in the mathematical model is introduced using the reaction. Recently, some new models have been proposed, where the memory phenomenon is taken into account by changing the Fick's law, see, e.g., [1, 2, 6].

On the context of biological phenomena, the qualitative properties of the solution of the nonlinear problem (1.1)-(1.3) have an important role on the description of the dynamic of the species that are being studied. Such qualitative properties depend on the behaviour of reaction terms.

It is known that in general the explicit expression for the solution of (1.1)-(1.3) is unavailable, numerical methods are the only way to get quantitative information to the nonlinear problem (1.1)-(1.3). The study of delay Cauchy or delay IBVPs has been very fruitful in the last twenty years, see, e.g., the books [3, 4, 14, 15] and the references therein. Moreover, the study of mathematical models containing delay equations continues to be a fruitful topic. We mention, without being exhaustive, the papers [7, 10, 12] contain the analysis of some biological systems, [9] presents a qualitative study of the solution of a hyperbolic delay equation. In [8], spectral collocation methods for a parabolic reaction-diffusion equation of type (1.1) are studied.

The characterization of the behaviour of the solution u of the IBVP (1.1)-(1.3) and the solution u_h^n of its discretization using the behaviour of the reaction term f is the aim of this paper. This characterization has an important role on the description of the behaviour of whole system.

Using energy method we establish estimates for u and u_h^n that depend on the derivatives of the reaction term f. As a consequence of these estimates, we reach conclusions concerning the stability of the solutions when the initial condition u_0 is perturbed.

The paper is organized as follows. In Section 2 we consider IBVPs (1.1)-(1.3) with the reaction term depending only on $u(x, t-\tau)$. In Section 2.1 the behaviour of the solution u and its stability are studied. In Section 2.2 a numerical method which can be seen as a combination of the spatial discretization defined by the centered finite difference operators and a time integration defined by the θ-method is considered. We study the behaviour of the finite difference solution and a discrete version of the result established in the continuous context is obtained. The stability and the convergence of the numerical method are also proved. The procedures used for the continuous and discrete models with a reaction term depending on $u(x, t-\tau)$ are