Journal of Computational Mathematics, Vol.26, No.5, 2008, 633–656.

STRONG STABILITY PRESERVING PROPERTY OF THE DEFERRED CORRECTION TIME DISCRETIZATION*

Yuan Liu[1)]
Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
Email: liuyuan@mail.ustc.edu.cn
Chi-Wang Shu
Division of Applied Mathematics, Brown University, Providence, RI 02912, USA
Email: shu@dam.brown.edu
Mengping Zhang
Department of Mathematics, University of Science and Technology of China, Hefei 230026, China
Email: mpzhang@ustc.edu.cn

Abstract

In this paper, we study the strong stability preserving (SSP) property of a class of deferred correction time discretization methods, for solving the method-of-lines schemes approximating hyperbolic partial differential equations.

Mathematics subject classification: 65L06.
Key words: Strong stability preserving, Deferred correction time discretization.

1. Introduction

In this paper, we are interested in the numerical solutions of hyperbolic partial differential equations (PDEs). A typical example is the nonlinear conservation law

$$u_t = -f(u)_x. \tag{1.1}$$

A commonly used approach to design numerical schemes for approximating such PDEs is to first design a stable spatial discretization, obtaining the following method-of-lines ordinary differential equation (ODE) system,

$$u_t = L(u), \tag{1.2}$$

to approximate (1.1). Notice that even though we use the same letter u in (1.1) and (1.2), they have different meanings. In (1.1), $u = u(x,t)$ is a function of x and t, while in (1.2), $u = u(t)$ is a (vector) function of t only. Stable spatial discretization for (1.1) includes, for example, the total variation diminishing (TVD) methods [6], the weighted essentially non-oscillatory (WENO) methods [7], and the discontinuous Galerkin (DG) methods [1]. In this paper, we *assume* that the spatial discretization (1.2) is stable for the first-order Euler forward time discretization

$$u^{n+1} = u^n + \Delta t L(u^n) \tag{1.3}$$

* Received June 8, 2007 / Revised version received September 20, 2007 / Accepted October 15, 2007 /
1) Current address: Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556-4618, USA.

under a suitable time step restriction

$$\Delta t \le \Delta t_0. \tag{1.4}$$

This stability is given as

$$\|u^{n+1}\| \le \|u^n\| \tag{1.5}$$

for a suitable norm or semi-norm $\|\cdot\|$. For the TVD schemes [6], $\|\cdot\|$ is taken as the total variation semi-norm. For technical reasons, we would also need a different but closely related spatial discretization to (1.1):

$$u_t = \tilde{L}(u) \tag{1.6}$$

with the property that the first-order "backward" time discretization

$$u^{n+1} = u^n - \Delta t \tilde{L}(u^n) \tag{1.7}$$

is stable in the sense of (1.5) under the same time step restriction (1.4). For the conservation law (1.1), the operator $\tilde{L}$ can often be obtained simply by reversing the wind direction in the upwind approximation. We refer to, e.g., [1, 7, 11] for such implementation in ENO, WENO and DG methods.

Even though the fully discretized scheme (1.3) is assumed to be stable as in (1.5), it is only first-order accurate in time. For a high-order spatial discretization such as in the WENO and DG methods, we would certainly hope to have higher-order accuracy in time as well. A higher-order time discretization for (1.2) is called strong stability preserving (SSP) with a CFL coefficient c, if it is stable in the sense of (1.5) under a possibly modified time step restriction

$$\Delta t \le c\,\Delta t_0. \tag{1.8}$$

SSP time discretizations were first developed in [10] for multi-step methods and in [11] for Runge-Kutta methods. They were referred to as TVD time discretizations in these papers, since the semi-norm involved in the stability (1.5) was the total variation semi-norm. More general SSP time discretizations can be found in, e.g., [3, 4, 12, 13]. The review paper [5] summarizes the development of the SSP method until the time of its publication.

In this paper we study the SSP property of a newly developed time discretization technique, namely the (spectral) deferred correction (DC) method constructed in [2]. An advantage of this method is that it is a one step method (namely, to march to time level $n+1$ one would only need to store the value of the solution at time level n) and can be constructed easily and systematically for any order of accuracy. This is in contrast to Runge-Kutta methods which are more difficult to construct for higher order of accuracy, and to multi-step methods which need more storage space and are more difficult to restart with a different choice of the time step Δt. Linear stability, such as the A-stability, $A(\alpha)$-stability, or L-stability issues for the DC methods were studied in, e.g., [2, 8, 14]. However, for approximating hyperbolic equations such as (1.1) with discontinuous solutions, linear stability may not be enough and one would hope the time discretization to have the SSP property as well.

The $(s+1)$-th order DC time discretization to (1.2) that we consider in this paper can be formulated as follows. We first divide the time step $[t^n, t^{n+1}]$, where

$$t^{n+1} = t^n + \Delta t$$

into s subintervals by choosing the points $t^{(m)}$ for $m = 0, 1, \cdots, s$ such that

$$t^n = t^{(0)} < t^{(1)} < \cdots < t^{(m)} < \cdots < t^{(s)} = t^{n+1}.$$