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SUPERCONVERGENCE ANALYSIS OF FINITE ELEMENT METHODS FOR OPTIMAL CONTROL PROBLEMS OF THE STATIONARY BÉNARD TYPE*

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Abstract

In this paper, we consider the finite element approximation of the distributed optimal control problems of the stationary Bénard type under the pointwise control constraint. The states and the co-states are approximated by polynomial functions of lowest-order mixed finite element space or piecewise linear functions and the control is approximated by piecewise constant functions. We give the superconvergence analysis for the control; it is proved that the approximation has a second-order rate of convergence. We further give the superconvergence analysis for the states and the co-states. Then we derive error estimates in L^∞-norm and optimal error estimates in L^2-norm.

Mathematics subject classification: 49J20, 65N30.
Key words: Optimal control problem, The stationary Bénard problem, Nonlinear coupled system, Finite element approximation, Superconvergence.

1. Introduction

The control of viscous flow for the purpose of achieving some desired objective is crucial to many technological and scientific applications. The Boussinesq approximation of the Navier-Stokes system is frequently used as mathematical model for fluid flow in semiconductor melts. In many crystal growth technics, such as Czochralski growth and zone-melting technics, the behavior of the flow has considerable impact on the crystal quality. It is therefore quite natural to establish flow conditions that guarantee desired crystal properties. As control actions, they include distributed forcing, distributed heating, and others. For example, the control of vorticity has significant applications in science and engineering such as control of turbulence and control of crystal growth process.

Considerable progress has been made in mathematics, physics and computation of the optimal control problems for the viscous flow; see [1, 2, 9, 12, 14, 15] and references therein. Optimal control problems for the thermally coupled incompressible Navier-Stokes equation by Neumann and Dirichlet boundary heat controls were considered in [12, 15]. Also, the time dependent problems were considered in the literature. In this article, we consider the Bénard problem whose state is governed by the Boussinesq equations, which is crucial to many technological and scientific applications. Without the control constraint, the analysis of approximation about optimal control of the stationary Bénard problem was considered in [20], and it uses the

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gradient iterative method to solve the discretized equations. For the constrained control case, there seems to be little work on this problem. This paper is concerned with the finite element approximation and error analysis of the constrained optimal control problem of the stationary Bénard problem:

$$(\mathcal{P}) \quad \min_{Q\in K} \quad J(Q) = \{\frac{1}{2}\|\mathbf{u}-\mathbf{U}\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|Q\|^2_{0,\Omega}\},$$

subject to the Boussinesq system:

$$\begin{aligned} &(a) \quad -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = T\mathbf{g} + f \quad in\ \Omega,\\ &(b) \quad \nabla\cdot\mathbf{u} = 0 \quad in\ \Omega,\\ &(c) \quad -\kappa\Delta T + \mathbf{u}\cdot\nabla T = Q \quad in\ \Omega,\\ &(d) \quad \mathbf{u} = 0 \quad T = 0 \quad on\ \partial\Omega \end{aligned} \tag{1.1}$$

and subject to the control constraint

$$K = \{Q \in L^2(\Omega) : Q(x) \geq 0;\ a.e.\ x \in \Omega\}, \tag{1.2}$$

where Ω is a regular bounded and convex open set in $\mathbb{R}^n$ ($n = 2$, or 3), with $\partial\Omega \in C^{1,1}$. $\mathbf{u}, p, T$ denote the velocity, pressure and temperature fields, respectively, f is a body force, and control Q. The vector $\mathbf{g}$ is in the direction of gravitational acceleration and $\kappa > 0$ the thermal conductivity parameter. In this paper we only consider, for the simplicity, the case where κ is constant. Assume $\nu > 0$ is the kinematic viscosity.

The optimal control problems $(\mathcal{P})$ are to seek the state variables $(\mathbf{u}, p, T)$ and Q such that the functional J is minimized subject to (1.1) where $\mathbf{U}$ is some desired velocity fields. The physics objective of the minimization problem is to match a desired flow field by adjusting the distributed control Q.

Approximation properties of the optimal control problems have long been investigated in the past years. For some classic work, we refer to Falk [10], Geveci [11] and Malanowski [26]. Theory and numerical results for elliptic control problems have been known for a long time, and can be found, for example, in Casas, Mateos, and Tröltzsch [5] or Casas and Tröltzsch [7], [24] and [25]. However, new discretization concepts have been developed in recent years. The variational approach by Hinze [16] and the superconvergence approach of Meyer and Rösch [27] can achieve approximation order h^2 in the L^2-norm using the piecewise constant control approximation for some simpler linear optimal control problems. However there seems to exist few known result on the analysis of the above control problem, which is a coupled nonlinear control problem.

In this work we show that the method cited above can be adapted to the Boussinesq equations. Here the control is discretized by piecewise constant functions. Clearly, the optimal approximation order of the control is expected to be h. However, we will show a superconvergence result that improves the order to h^2 only assuming first order global regularity. We will show the state $\mathbf{u}$ and the related co-state have the approximation order of h^2 in the L^2-norm.

The paper is organized as follows. In Section 2, we give some notations and assumptions that will be used throughout the paper. In Section 3, we will discuss the finite element approximation of the optimal control problem. In Section 4, the main results will be given and the proof of the superconvergence results will be presented in Section 5.