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THE OPTIMAL CONVERGENCE ORDER OF THE DISCONTINUOUS FINITE ELEMENT METHODS FOR FIRST ORDER HYPERBOLIC SYSTEMS*

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Abstract

In this paper, a discontinuous finite element method for the positive and symmetric, first-order hyperbolic systems (steady and nonsteady state) is constructed and analyzed by using linear triangle elements, and the $\mathcal{O}(h^2)$-order optimal error estimates are derived under the assumption of strongly regular triangulation and the H^3-regularity for the exact solutions. The convergence analysis is based on some superclose estimates of the interpolation approximation. Finally, we discuss the Maxwell equations in a two-dimensional domain, and numerical experiments are given to validate the theoretical results.

Mathematics subject classification: 65N30, 65M60.
Key words: First order hyperbolic systems, Discontinuous finite element method, Convergence order estimate.

1. Introduction

It is well known that for the k-th order finite element approximations to elliptic or parabolic problems, the optimal order error estimate in the L_2 norm is of order $\mathcal{O}(h^{k+1})$ with the exact solution u in $H^{k+1}(\Omega)$. However, for linear hyperbolic problems, it is still a completely unsolved problem whether or not the finite element solutions admit this optimal order estimate. Generally speaking, the convergence order of the Galerkin finite element method for hyperbolic problems is of order $\mathcal{O}(h^k)$, that is one order lower than the approximation order of the finite element space; see, e.g., [7,14]. In addition, in [7] Dupont gave a counterexample by using a third-order Hermite element to indicate that this convergence rate is sharp. Since then, in order to obtain the high accuracy and cope with the characteristics of hyperbolic problems, the discontinuous Galerkin method is proposed and used extensively in this area; see, e.g., [4,9,12,15,18,20].

Historically, the original discontinuous Galerkin finite element method was introduced by Reed and Hill [18] in 1973 to solve the linear neutron transport equation. Soon Lesaint and Raviart [15] gave its mathematical analysis and obtained the $\mathcal{O}(h^k)$-order error estimates when the k-th order discontinuous finite element spaces were used. Later on, Johnson and Pitkaranta [12] improved this convergence rate to $\mathcal{O}(h^{k+\frac{1}{2}})$, and Peterson [17] further proved that, under the quasi-uniform triangulation condition, the $\mathcal{O}(h^{k+\frac{1}{2}})$ convergence rate is sharp, namely, this is the optimal order error estimate for discontinuous Galerkin finite element approximations to first-order hyperbolic problems. On the superconvergence research, Lesaint and Raviart [15] first obtained estimate of the form $\|u - u_h\| \le Ch^{k+1}\|u\|_{k+2}$, for rectangular mesh finite elements (also see [16] for the piecewise constant approximation case), and Richter [19] did so

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for semi-uniform triangular meshes under the curious assumption that all element edges are bounded away from the characteristic direction of the hyperbolic equation, which is less significant in the practical case. However, all the papers mentioned above discuss the single scalar hyperbolic equation only. Indeed, the discontinuous Galerkin finite element methods have been extended successfully to linear systems of first-order hyperbolic equations and nonlinear problems. Now there is a lot of literature available in this area. For example, Johnson and Huang [13] studied this method for Friedrichs system of equations, and gained the $\mathcal{O}(h^{k+\frac{1}{2}})$-order error estimate, this result was extended to the initial-boundary value problems of positive and symmetric, linear systems of hyperbolic equations by Zhang in [20]. Huang also discussed discontinuous Galerkin finite element methods for mixed Tricomi equations and nonlinear vorticity transport equations [9,10]. Since the 1990s, Cockburn and Shu et al. systematically studied the discontinuous Galerkin finite element method for nonlinear convection laws and related problems. By using numerical flow of finite differences with higher resolution, TVB, and gradient limiters, some new type of discontinuous Galerkin finite element methods for various hyperbolic problems were designed, see, e.g., [1,2,3,4]. Furthermore, the Maxwell equations with periodic boundary conditions were also discussed by using the locally divergence-free discontinuous Galerkin method in [5]. For more literature, the reader is referred to Cockburn and Shu's review article [3] and the references therein.

In this paper, we will discuss the discontinuous linear finite element approximations to positive and symmetric linear hyperbolic systems (steady and nonsteady state). Under the assumption of strongly regular triangulation and H^3-regularity for the exact solutions, the $\mathcal{O}(h^2)$-order optimal error estimate is established. The theoretical tools for the error analysis are some superclose estimates of interpolation approximation that are also derived in this paper. In our discontinuous finite element method (see (2.7)-(2.8)), the approximations of the traces of the fluxes on the boundary of the elements (the so-called numerical fluxes) are different from that introduced by Reed and Hill [18] or Lesaint and Raviart [15] in the original DG method. Generally speaking, our method will lead to an implicit scheme, while those schemes in [15,18] are in an explicit fashion such that the discrete equations can be solved explicitly through an ordering, element by element. The advantage of our method is that it allows us to derive the optimal order error estimates. To the authors' knowledge, very few optimal order error estimates have been obtained for hyperbolic problems, even in one dimensional case. Hence, our research work in this paper is theoretically significant.

Let $\Omega \subset R^2$ be a polygonal domain, $J_h = \{e\}$ the finite element triangulation of the domain Ω parameterized by the mesh size h so that $\overline{\Omega} = \cup_{e\in J_h}\{\,\overline{e}\,\}$. Introduce the discontinuous linear finite element space S_h defined by

$$S_h = \{\, v \in L_2(\Omega) : v|_e \text{ is linear, } \forall e \in J_h \,\}.$$

We will use the standard notations for the Sobolev spaces $W_p^m(\Omega)$ with corresponding norms and seminorms, and when $p = 2$, $W_2^m(\Omega) = H^m(\Omega)$, $\|\cdot\|_{m,2} = \|\cdot\|_m$. Denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the standard inner product and norm in $L_2(\Omega)$. Let X be a Banach space. For constant $T > 0$, we will also use the space,

$$L_p(0,T;X) = \Big\{\, v(t) : (0,T) \to X : \ \|v\|_{L_p(X)} = \Big(\int_0^T \|v(t)\|_X^p \, dt \Big)^{\frac{1}{p}} < \infty \Big\}.$$

In this paper, the letter C represents a generic constant independent of the mesh size h.

The plan of this paper is as follows. In Section 2, the discontinuous finite element approximations are constructed for steady and nonsteady positive and symmetric hyperbolic systems,