Journal of Computational Mathematics, Vol.26, No.5, 2008, 716–727.

UNIFORMLY A POSTERIORI ERROR ESTIMATE FOR THE FINITE ELEMENT METHOD TO A MODEL PARAMETER DEPENDENT PROBLEM*

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Abstract

This paper proposes a reliable and efficient a posteriori error estimator for the finite element methods for the beam problem. It is proved that the error can be bounded by the computable error estimator from above and below up to multiplicative constants that do neither depend on the meshsize nor on the thickness of the beam.

Mathematics subject classification: 65N30, 65N15.
Key words: The beam problem, A posteriori error estimator, Finite element method.

1. Introduction

The beam model [1,2,12] considered here reads: Seek two functions $\varphi_d(x)$ and $\omega_d(x)$ defined in the unit interval $I = [0, 1]$ such that

$$\begin{aligned} &-\varphi_d'' + d^{-2}(\varphi_d - \omega_d') = 0, && \text{in } (0,1), \\ &d^{-2}(\varphi_d - \omega_d')' = g, && \text{in } (0,1), \\ &\varphi_d(0) = \varphi_d(1) = \omega_d(0) = \omega_d(1) = 0. \end{aligned} \tag{1.1}$$

Here and throughout the paper, the parameter d $(0 < d < 1)$ denotes the thickness of the beam. This model may be derived from the equations of plane linear elasticity by dimensional reduction, which means that an undisplaced plane body occupying the region $\{0 \le x \le 1, -\frac{d}{2} \le y \le \frac{d}{2}\}$ be subject to a smooth vertical body force $-d^2 g(x)$. Physically ω_d represents the vertical displacement of the midline, and φ_d the rotation of the cross section.

The corresponding variational formulation is as follows. Given $g \in L^2(I)$, find φ_d, $\omega_d \in H_0^1(I)$ such that

$$(\varphi_d', \psi') + d^{-2}(\varphi_d - w_d', \psi - \upsilon') = (g, \upsilon), \quad \text{for all } \psi, \upsilon \in H_0^1(I), \tag{1.2}$$

with the shear force

$$\gamma_d = d^{-2}(\varphi_d - \omega_d'). \tag{1.3}$$

This paper is devoted to this beam problem which is difficult due to the small parameter related to the thickness of the beam. For the reason of a highly desirable quality of a numerical method, we hope to approximate the solution as accurately as possible for all values of this parameter. For a priori error estimate analysis of the beam problem, Arnold [1,2] investigates the robustness of two families of finite element methods with respect to the parameter d. He points out that a standard linear finite element is found to be not robust at all which

* Received August 31, 2007 / Revised version received November 28, 2007 / Accepted December 4, 2007 /

means the approximation errors do not converge to zero at the optimal rates uniformly in d. Although the same method does converge uniformly with respect to the parameter when the spaces of piecewise polynomials of order at least two are used, the approximation degenerates as the thickness of the beam decreases, resulting in a reduced uniform order of convergence. Comparatively, the mixed method he considers as the second method produces good results and the errors converge uniformly with respect to the parameter for (almost) any choice of finite element spaces for the original displacement without the degeneracy mentioned above. All aforementioned papers are only concerning the a priori error analysis of the beam problem. As for a posteriori error analysis of this problem, as far as we know, no work can be found in the literature.

It is worth mentioning that recently there are some progress on the a posteriori error estimates of the Reissner-Mindlin plate problem [7,8,11] and the unifying theory of a posteriori error analysis of finite element methods [5–9,11]. In [11], Hu and Huang introduce a sparse mixed formulation and establish the equivalence between the energy norms of errors and the dual norms of the residuals. They propose some sufficient conditions and provide a unified framework for the a posteriori error analysis of the finite element methods of the Reissner-Mindlin plate problem. This paper follows these ideas and establishes some residual representation which is closely related to the approximation errors, and presents a posteriori error estimates of the beam problem. Then we analyze the Arnold's discrete scheme of this problem [1,2] within this framework, and propose a reliable and efficient residual-based a posteriori error estimator. The related multiplicative constants do neither depend on the meshsize nor on the beam thickness.

The outline of the paper is as follows. In Section 2 we establish some equivalence between the norms of errors and the dual norms of some residuals. In Section 3 we present Arnold's discrete scheme for the beam problem. The main results of this paper will be also stated. We prove the results in Sections 4 and 5.

Throughout this paper, all function spaces will be formed with respect to the unit interval $I = [0, 1]$. For functions $f(x)$ and $g(x)$ defined in $[0, 1]$, we let (f, g) denote the inner productor $\int_0^1 f(x)g(x)dx$. The associated L^2-norm of the function is written as $\|f\|$, while $\|f\|_r$ denote the norm in the Sobolev space $H^r(I) : \|f\|_r^2 = \|f\|^2 + \|f^{'}\|^2 \ldots + \|f^{(r)}\|^2$, where $f^{(r)} = \frac{d^r f}{dx^r}$. The space $H_0^1(I) = \{f \in H^1(I) | f(0) = f(1) = 0\}$, on which the norm $\|f^{'}\|$ is equivalent to the H^1 norm. The space $H^{-1}(I)$ is dual to $H_0^1(I)$ equipped with the norm

$$\|g\|_{-1} = \sup_{f \in H_0^1} \frac{(f, g)}{\|f'\|}, \qquad \forall g \in H^{-1}(I).$$

In this paper the generic constant $C > 0$ independent of the beam thickness d below may be different at different occurrences. An inequality $a \preceq b$ replaces $a \leq Cb$, $a \approx b$ abbreviates $a \preceq b \preceq a$.

2. Residual-based a Posteriori Error Control

Follows the ideas of [3,7,11], let $d^{-2} = \frac{3}{4} + \beta^{-2}$ and introduce an additional independent variable

$$\gamma_d^* = \beta^{-2}(\varphi_d - \omega_d'). \tag{2.1}$$

We obtain a new established mixed version of the beam problem which is equivalent to the weak formulation (1.2): Given $g \in L^2(I)$, find $(\varphi_d, \omega_d, \gamma_d^*) \in W \times \Theta \times Q = H_0^1(I) \times H_0^1(I) \times L^2(I)$