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CONVERGENCE OF A MIXED FINITE ELEMENT FOR THE STOKES PROBLEM ON ANISOTROPIC MESHES*

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Abstract

The main aim of this paper is to study the convergence properties of a low order mixed finite element for the Stokes problem under anisotropic meshes. We discuss the anisotropic convergence and superconvergence independent of the aspect ratio. Without the shape regularity assumption and inverse assumption on the meshes, the optimal error estimates and natural superconvergence at central points are obtained. The global superconvergence for the gradient of the velocity and the pressure is derived with the aid of a suitable postprocessing method. Furthermore, we develop a simple method to obtain the superclose properties which improves the results of the previous works.

Mathematics subject classification: 65N30.
Key words: Mixed finite element, Stokes problem, Anisotropic meshes, Superconvergence, Shape regularity assumption and inverse assumption.

1. Introduction

There have been many studies for the mixed finite elements approximation to the stationary Stokes problem [10, 15, 16, 21, 25] which satisfy the Babuška-Brezzi condition (see, e.g., [5, 11]). The optimal error estimates were obtained under the shape regularity assumption [9, 14] on the meshes. However, the solution of the Stokes problem may have anisotropic behavior in parts of the domain, for instance, the presence of boundary layers and other localized features. This means that the solution varies significantly in certain directions with less significant changes along the other ones. It is an obvious idea to reflect this anisotropy in the discretization by using anisotropic meshes with a small mesh size in the direction of the rapid variation of the solution and a larger mesh size in the other direction, where elements are aligned to follow (in some sense) the geometry of the solution. Compared with the standard isotropic techniques, the number of degrees of freedom required for a given accuracy may be considerably reduced.

Recently, some efforts have been made to develop stable mixed methods for the meshes that include elements of arbitrary high aspect ratio. For instance, Schötzau *et al.* [23, 24] for $\mathcal{Q}_{k+1}-\mathcal{Q}_{k-1}$ families, Becker and Rannacher [6, 7] for $\mathcal{Q}_1-\mathcal{Q}_0$, $\mathcal{Q}_1-\mathcal{Q}_1$, Apel and Nicaise [3] for $\tilde{\mathcal{Q}}_1-\mathcal{Q}_0$. By $\mathcal{Q}_1$ we denote, as usual, the space of bilinear functions, and by $\tilde{\mathcal{Q}}_1$ the rotated

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$\mathcal{Q}_1$ element. All the methods developed in the above works used high aspect ratio meshes, although most of them placed some restrictions on the meshes. For the stability of the method it is required that the discrete spaces satisfy the Babuška-Brezzi condition with a constant (inf-sup constant) independent of the aspect ratio of the elements. It has been reported by Russo that the mini element becomes instable on anisotropic meshes (cf. [22]). As to the estimate of the interpolation error under anisotropic meshes, Apel [1, 4, pp.35-38] presented a criterion to judge the anisotropy of an interpolation; Chen *et al.* developed an anisotropic interpolation theorem in [12, 13, 27, 28], which is much easier to use. In this work, we consider another familiar scheme which can be regarded as a low order Bernardi-Raugel element (cf. [8]) under anisotropic rectangular meshes. Recently, the stability of this scheme with the inf-sup constant independent of the aspect ratio has been discussed by Apel and Nicaise in [2]. We check the anisotropy of the interpolation of velocity, and then the optimal error estimates can be obtained by using the anisotropic interpolation theorem.

On the other hand, the superconvergence for the mixed elements is very effective in practice. Some superconvergence results for several mixed finite elements have been obtained when the meshes are sufficiently good. Lin and Pan in [20] and [18] proved $\mathcal{O}(h^2)$-superconvergence for the $\mathcal{Q}_1 - \mathcal{Q}_0$ element under square meshes and $\mathcal{O}(h^3)$-superconvergence for the biquadratic-linear element over uniform rectangular meshes, respectively. On quasi-uniform rectangular meshes, the $\mathcal{O}(h^2)$-superconvergence for the Bernardi-Raugel element was obtained by [18]. A key concept in their derivation is the integral identity technique which has been proven to be an efficient tool for the superconvergence analysis of rectangular finite elements (cf. [17, 19]). In this paper, a simple method is developed to obtain the superclose results. The basic tool employed by us is the well-known Bramble-Hilbert Lemma. Furthermore, compared with the previous works, our results can be worked without the shape regularity assumption and inverse assumption requirement on the meshes and can be applied to more general meshes.

The paper is organized as follows: we investigate the anisotropic interpolation properties of the Bernardi-Raugel element in Section 2. In Section 3, based on the stability of this scheme with the inf-sup constant independent of the aspect ratio, which has been obtained in [2], we get the optimal anisotropic error estimates. Without the shape regularity assumption and inverse assumption requirement on the meshes, the superclose result and global $\mathcal{O}(h^2)$-superconvergence of the Bernardi-Raugel element are obtained under rectangular meshes in Section 4 and Section 5, respectively. Finally, natural superconvergence at central points is derived in Section 6.

2. Some Notations and Basic Estimates

In this section, we introduce some notations and recall some estimates that are basic for our subsequent arguments.

For the sake of convenience, let $\Omega \subset \mathbf{R}^2$ be a convex polygon composed by a family of rectangular meshes J_h which need not satisfy the shape regular conditions. $\forall K \in J_h$, we denote the barycenter of the element K by (x_K, y_K), the length of edges parallel to x-axis and y-axis by $2h_{K1}, 2h_{K2}$ respectively, $h_K = \max\{h_{K1}, h_{K2}\}, h = \max_{K\in\mathcal{J}_h} h_K$, $h_K^\alpha = h_{K1}^{\alpha_1} h_{K2}^{\alpha_2}$. Assume that $\hat{K} = [-1,1] \times [-1,1]$ is the reference element, the four vertices are: $\hat{a}_1 = (-1,-1), \hat{a}_2 = (1,-1), \hat{a}_3 = (1,1), \hat{a}_4 = (-1,1)$, and its 4 sides are $\hat{l}_1 = \hat{a}_1\hat{a}_2, \hat{l}_2 = \hat{a}_2\hat{a}_3, \hat{l}_3 = \hat{a}_3\hat{a}_4, \hat{l}_4 = \hat{a}_4\hat{a}_1$. Then there exists a unique mapping $F_K : \hat{K} \to K$ defined as

$$\begin{cases} x = x_K + h_{K1}\xi, \\ y = y_K + h_{K2}\eta. \end{cases} \tag{2.1}$$