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THE SECOND-ORDER OPTIMALITY CONDITIONS FOR VARIABLE PROGRAMMING*

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Abstract

We study in this paper the continuity of the objective function for variable programming. In particular, we study the second-order optimality conditions for unconstrained and constrained variable programming. Some new second-order sufficient and necessary conditions are obtained.

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Key words: Second-order optimality condition, Unconstrained, Constrained, Variable programming.

1. Introduction

Consider the unconstrained variable programming problem (VPI)

$$\min_{x\in R^n}\max_{i\in I(x)} f_i(x), \tag{1.1}$$

where

$$I(x)=\{j\in K|q_j(x)=q(x)\}, \tag{1.2a}$$
$$q(x)=\max_{l\in K}\{q_l(x)\},\quad K=\{1,2,\cdots,k\}. \tag{1.2b}$$

We also consider the constrained variable programming problem (VPII)

$$\min_{x}\max_{i\in I(x)} f_i(x) \tag{1.3}$$
$$s.t\quad c_j(x)\le 0,\qquad j=1,2,\cdots,p, \tag{1.4}$$

where

$$I(x)=\{j\in K|q_j(x)=\max_{l\in K}q_l(x)\},\quad K=\{1,2,\cdots,k\}. \tag{1.5}$$

In [8], Wang and Xu gave some theoretical results for the optimality conditions. In [3,4], Jiao et al. presented some useful theories and algorithms for (1.1)-(1.2) and (1.3)-(1.5). However, these theoretical results are only first-order optimality conditions. In this paper, we focus on the second-order optimality conditions for unconstrained and constrained variable programming.

Let

$$\varphi(x)=\max_{i\in I(x)} f_i(x). \tag{1.6}$$

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Fixing x, let us consider the set of indices $R(x)$ defined by

$$R(x) = \{i | i \in I(x), f_i(x) = \varphi(x)\}. \tag{1.7}$$

Lemma 1.1 ([3]) *For $x_0 \in R^n$, suppose that the functions $q_i(x), i \in K$, are continuous at point x_0, then there exists a real number $\delta > 0$ such that for all $x \in S(x_0, \delta) := \{x | ||x - x_0|| < \delta\})$,*

$$I(x) \subseteq I(x_0). \tag{1.8}$$

Lemma 1.2 ([3]) *For $x_0 \in R^n$, let the functions $f_i(x), q_i(x), i \in K$, be continuous at point x_0. If there exists a real number $\delta > 0$ such that for all $x \in S(x_0, \delta)$,*

$$I(x) \cap R(x_0) \neq \emptyset, \tag{1.9}$$

then

$$\varphi(x) = \max_{i \in I(x)} f_i(x) = \max_{i \in I(x) \cap R(x_0)} f_i(x). \tag{1.10}$$

Theorem 1.1. *For $x_0 \in R^n$, suppose that the functions $f_i(x), q_i(x), i \in K$, are continuous at point x_0. Then $\varphi(x)$ is continuous at point x_0 if and only if there exists a real number $\delta > 0$ such that for all $x \in S(x_0, \delta)$,*

$$I(x) \cap R(x_0) \neq \emptyset. \tag{1.11}$$

Proof. If $I(x) \cap R(x_0) \neq \emptyset$, we obtain from Lemma 1.2 that

$$\lim_{x \to x_0} \varphi(x) = \lim_{x \to x_0} \max_{i \in I(x) \cap R(x_0)} f_i(x) = \max_{i \in R(x_0)} f_i(x_0) = \varphi(x_0).$$

Hence, $\varphi(x)$ is continuous at point x_0. On the other hand, suppose that $\varphi(x)$ is continuous at point x_0. If there exists a sequence $x_i \to x_0$ such that $I(x_i) \cap R(x_0) = \emptyset$, then for $\forall \epsilon$ satisfying $0 < \epsilon \leq \frac{1}{2}(\varphi(x_0) - f_{j_0}(x_0))$, where

$$j_0 \in \Big\{ j | f_j(x_0) = \max_{j \in \{\lim\limits_{x_i \to x_0} I(x_i)\}} \{f_j(x)\} \Big\},$$

there exists an integer N_0 such that for $i > N_0$,

$$\varphi(x_i) = \max_{j \in I(x_i)} \{f_j(x_i)\} \leq f_{j_0}(x_0) + \epsilon.$$

Thus,

$$\begin{aligned} |\varphi(x_i) - \varphi(x_0)| &\geq |f_{j_0}(x_0) + \epsilon - \varphi(x_0)| \\ &\geq |f_{j_0}(x_0) - \varphi(x_0)| - \epsilon \\ &\geq \frac{1}{2}(\varphi(x_0) - f_{j_0}(x_0)), \end{aligned}$$

which is a contradiction with the assumption that $\varphi(x)$ is continuous at point x_0. Hence, the theorem is proved. □

For $x_0 \in R^n$, and $\forall h \in R^n$, $R'(x_0, h)$ is defined by

$$R'(x_0, h) = \lim_{\alpha \to 0^+} I(x_0 + \alpha h) \cap R(x_0), \tag{1.12}$$

$$R'(x_0) = \cup_{||h||=1} R'(x_0, h). \tag{1.13}$$

Furthermore, let

$$L(x) = \Big\{ z = \sum_{i \in R'(x_0)} \mu_i \bigtriangledown f_i(x_0) | \mu_i \geq 0, i \in R'(x_0), \sum_{i \in R'(x_0)} \mu_i = 1 \Big\}. \tag{1.14}$$