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CONVERGENCE ANALYSIS ON ITERATIVE METHODS FOR SEMIDEFINITE SYSTEMS*

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Abstract

The convergence analysis on the general iterative methods for the symmetric and positive semidefinite problems is presented in this paper. First, formulated are refined necessary and sufficient conditions for the energy norm convergence for iterative methods. Some illustrative examples for the conditions are also provided. The sharp convergence rate identity for the Gauss-Seidel method for the semidefinite system is obtained relying only on the pure matrix manipulations which guides us to obtain the convergence rate identity for the general successive subspace correction methods. The convergence rate identity for the successive subspace correction methods is obtained under the new conditions that the local correction schemes possess the local energy norm convergence. A convergence rate estimate is then derived in terms of the exact subspace solvers and the parameters that appear in the conditions. The uniform convergence of multigrid method for a model problem is proved by the convergence rate identity. The work can be regraded as unified and simplified analysis on the convergence of iteration methods for semidefinite problems [8,9].

Mathematics subject classification: 65F10, 65N22, 65N55.
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1. Introduction

We consider the iterative methods for the following linear problem,

$$Au = b, \tag{1.1}$$

where A is a symmetric and positive semidefinite operator from V to V, V is a finite dimensional Hilbert space with the inner product $(\cdot,\cdot)$ and $b \in V$ is a vector in the range of A. Such semidefinite problems arise in many areas of applied mathematics. The finite element and/or

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finite difference discretizations of the Poisson equation with Neumann boundary conditions [2] and the linear elasticity equation with pure traction boundary conditions lead to such problems. Other more sophisticated examples can be found at the linear systems obtained from generalized finite element methods [15, 16], and the time dependent Navier-Stokes systems [3].

For such problems, in general, it is difficult to apply the direct methods in a straightforward manner (not to mention that direct methods are very expensive for large linear systems [23]). Iterative methods are desirable for large semidefinite systems and our focus in this paper will be made, in particular, on the convergence analysis of the classic iterative methods and the general subspace correction methods for semidefinte (singular) problems given in (1.1).

The studies of the classic iterative methods for singular systems and their convergence can be traced back to Keller, [7] and there have been many investigations by many researchers since then, see [1, 4, 8, 11] and also many references cited therein. The classical iterative methods discussed in those works are mainly based on a matrix splitting: $A = M - N$ and from the setting for the iterates $\{u^\ell\}_{\ell=0}$:

$$Mu^\ell = Nu^{\ell-1} + b \tag{1.2}$$

or equivalently,

$$M\big(u^\ell - u^{\ell-1}\big) = \big(b - Au^{\ell-1}\big). \tag{1.3}$$

All the convergence results require that the iterator M be an invertible matrix, except that in [8]. Furthermore, the setting in [8] requires $\mathcal{N}(M^t) \subset \mathcal{N}(A)$ which is necessary for the solvability of (1.2) for x^ℓ.

In this paper, we study iterative methods for (1.1) in the following form:

$$u^\ell = u^{\ell-1} + R\big(b - Au^{\ell-1}\big), \tag{1.4}$$

where R is a linear operator from V to V and it may be singular. We then present more refined necessary and sufficient conditions for the energy norm convergence of the iterative method (1.4). One advantage of such view is that no assumption on the null space is necessary.

The rest of paper will be devoted to establish a convergence rate identity for the general successive subspace correction method. The techniques of subspace correction methods are based on a "divide and conquer" strategy. Classic iterative methods as Gauss-Seidel method, and many multigrid and domain decomposition methods fall into this category of methods. Recently, authors provided a sharpest possible convergence estimate for the general subspace correction method for singular systems in a general Hilbert space setting, [9]. The current works are aimed to better understand the basic idea of obtaining the convergence rate estimate in a transparent manner restricting the problem in finite dimensional spaces.

The sharp convergence rate identity for the Gauss-Seidel method for the semidefinite system, is obtained relying only on the pure matrix manipulations. The idea will guide us to obtain the convergence rate identity for the general successive subspace correction methods. For the successive subspace correction methods, we assume that the local correction schemes possess the local energy norm convergence. We then derive a new version of the convergence rate identity [9] under minimal assumptions. We also get the convergence rate estimate in terms of the exact subspace solvers and the parameters that appear in the conditions. In Section 4, we give an example from an electrochemical model to illustrate how to apply our identity in designing an optimal multigrid method for such a singular system, and prove the uniform convergence for the multigrid method. As the results in this paper, we will be able to give the convergence criteria that are more refined and concise that that in [8, 9] and whose analysis becomes more transparent.