Journal of Computational Mathematics, Vol.27, No.1, 2009, 45–67.

THE l^1-STABILITY OF A HAMILTONIAN-PRESERVING SCHEME FOR THE LIOUVILLE EQUATION WITH DISCONTINUOUS POTENTIALS*

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Abstract

We study the l^1-stability of a Hamiltonian-preserving scheme, developed in [Jin and Wen, *Comm. Math. Sci.,* 3 (2005), 285-315], for the Liouville equation with a discontinuous potential in one space dimension. We prove that, for suitable initial data, the scheme is stable in the l^1-norm under a hyperbolic CFL condition which is in consistent with the l^1-convergence results established in [Wen and Jin, *SIAM J. Numer. Anal.,* 46 (2008), 2688-2714] for the same scheme. The stability constant is shown to be independent of the computational time. We also provide a counter example to show that for other initial data, in particular, the measure-valued initial data, the numerical solution may become l^1-unstable.

Mathematics subject classification: 65M06, 65M12, 65M25, 35L45, 70H99.
Key words: Liouville equations, Hamiltonian preserving schemes, Discontinuous potentials, l^1-stability, Semiclassical limit.

1. Introduction

In [7], we constructed a class of numerical schemes for the d-dimensional Liouville equation in classical mechanics:

$$f_t + \mathbf{v} \cdot \nabla_{\mathbf{x}} f - \nabla_{\mathbf{x}} V \cdot \nabla_{\mathbf{v}} f = 0, \quad t > 0, \quad \mathbf{x}, \mathbf{v} \in R^d, \tag{1.1}$$

where $f(t, \mathbf{x}, \mathbf{v})$ is the density distribution of a classical particle at position $\mathbf{x}$, time t and traveling with velocity $\mathbf{v}$. $V(\mathbf{x})$ is the potential. The main interest is in the case of a discontinuous potential $V(\mathbf{x})$, corresponding to a potential barrier. When V is discontinuous, the Liouville equation (1.1) is a linear hyperbolic equation with a measure-valued coefficient. One needs to provide additional condition in order to select a unique, physically relevant solution across the barrier. The main idea of the Hamiltonian-preserving schemes developed in [7] was to build into the numerical flux the particle behavior at the barrier. See also the related work on Hamiltonian-preserving schemes [2, 3, 5, 6, 8–13].

The Liouville equation (1.1) is a different formulation of Newton's second law:

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \quad \frac{d\mathbf{v}}{dt} = -\nabla_{\mathbf{x}} V, \tag{1.2}$$

* Received March 17, 2008 / accepted April 18, 2008 /

which is a Hamiltonian system with the Hamiltonian

$$H = \frac{1}{2}|\mathbf{v}|^2 + V(\mathbf{x}). \tag{1.3}$$

It is known from classical mechanics that the Hamiltonian remains constant across a potential barrier. By using this mechanism in the numerical flux, the schemes developed in [7] provide a physically relevant solution to the underlying problem. It was proved that the two schemes developed in [7], under a hyperbolic CFL condition, are positive, and stable under both l^∞ and l^1 norms in one space dimension except the l^1-stability of Scheme I. Scheme I uses a finite difference approach involving interpolations in the phase space and the l^1-stability of this scheme is more sophisticated. In this paper we consider this issue in details. We will prove that Scheme I is l^1-stable with the stability constant independent of the computational time if the initial data satisfy certain condition, but can be l^1-unstable if the initial data condition is violated. The initial data condition is satisfied when applying the decomposition technique proposed in [4] for solving the Liouville equation with measure-valued initial data arisen from the semiclassical limit of the linear Schrödinger equation. Recently the l^1-convergence of the same scheme under certain initial data condition has been established in [19] by applying the l^1-error estimates developed in [16, 18] for the immersed interface upwind scheme to the linear advection equations with piecewise constant coefficients. We show that the results established in this paper is in consistent with the convergence results established in [19] since the initial data condition considered in this paper is more general than that in [19].

The paper is organized as follows. In Sect. 2, we first present Scheme I developed in [7]. In Sect. 3, we prove the l^1-stability of this scheme for suitable initial data. We give a counter example in Sect. 4 to show that for more general initial data, in particular the measure-valued initial data, the numerical solution may become unbounded. We conclude the paper in Sect. 5.

2. A Hamiltonian-Preserving Scheme

Consider the Liouville equation in one space dimension:

$$f_t + \xi f_x - V_x f_\xi = 0 \tag{2.1}$$

with a discontinuous potential $V(x)$.

Without loss of generality, we employ a uniform mesh with grid points at $x_{i+\frac{1}{2}}, i = 0, \cdots, N$, in the x-direction and $\xi_{j+\frac{1}{2}}, j = 0, \cdots, M$ in the ξ-direction. The cells are centered at $(x_i, \xi_j), i = 1, \cdots, N, j = 1, \cdots, M$ with

$$x_i = \frac{1}{2}(x_{i+\frac{1}{2}} + x_{i-\frac{1}{2}}), \quad \xi_j = \frac{1}{2}(\xi_{j+\frac{1}{2}} + \xi_{j-\frac{1}{2}}).$$

The mesh size is denoted by $\Delta x = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \Delta\xi = \xi_{j+\frac{1}{2}} - \xi_{j-\frac{1}{2}}$. We also assume a uniform time step Δt and the discrete time is given by $0 = t_0 < t_1 < \cdots < t_L = T$. We introduce mesh ratios $\lambda_x^t = \Delta t/\Delta x, \lambda_\xi^t = \Delta t/\Delta\xi, \lambda_x^\xi = \Delta\xi/\Delta x$, assumed to be fixed. We define the cell averages of f as

$$f_{ij} = \frac{1}{\Delta x \Delta\xi} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{\xi_{j-\frac{1}{2}}}^{\xi_{j+\frac{1}{2}}} f(x, \xi, t)\, d\xi\, dx.$$

The 1-d average quantity $f_{i+1/2,j}$ is defined as

$$f_{i+1/2,j} = \frac{1}{\Delta\xi} \int_{\xi_{j-\frac{1}{2}}}^{\xi_{j+\frac{1}{2}}} f(x_{i+1/2}, \xi, t) d\xi\, .$$