Journal of Computational Mathematics, Vol.27, No.1, 2009, 68–88.

A POSTERIORI ERROR ESTIMATE FOR BOUNDARY CONTROL PROBLEMS GOVERNED BY THE PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract

In this paper, we discuss the a posteriori error estimate of the finite element approximation for the boundary control problems governed by the parabolic partial differential equations. Three different a posteriori error estimators are provided for the parabolic boundary control problems with the observations of the distributed state, the boundary state and the final state. It is proven that these estimators are reliable bounds of the finite element approximation errors, which can be used as the indicators of the mesh refinement in adaptive finite element methods.

Mathematics subject classification: 49J20, 65N30.
Key words: Boundary control problems, Finite element method, A posteriori error estimate, Parabolic partial differential equations.

1. Introduction

Finite element approximation plays a very important role in the numerical methods for optimal control problems. There have been extensive theoretical and numerical studies in this research direction. For instance, the error analysis for optimal control problems governed by linear elliptic equations has been established in [12, 13], the error estimates for some important flow control problems are given in [14], the error estimates for Dirichlet boundary control governed by semilinear elliptic equations are provided in [6]. Some recent progress in this area has been summarized in [24].

In recent years, the adaptive finite element method has been investigated extensively. It has become one of the most popular methods in the scientific computation and numerical modelling. Adaptive finite element approximation ensures a higher density of nodes in a certain area of the given domain, where the solution is more difficult to approximate, indicated by a posteriori error estimators. Hence it is among the most important means to boost the accuracy and efficiency of finite element discretizations. We acknowledge the pioneering work due to Babuška and Rheinboldt [2]. Further references can be found in the monographs [1, 3, 28], and the references therein.

Earlier works on a posteriori error estimates are concentrated on the elliptic partial differential equations. Later, there are many works about the a posteriori error estimates for parabolic problems. We mention the work of Eriksson and Johnson [10, 11], which is based on the analysis of linear dual problems of the corresponding error equations. The derived a posteriori error estimates depend on the H^2 regularity assumption on the underlying elliptic operator. In [25],

* Received September 14, 2007 / Revised version received January 24, 2008 / Accepted March 21, 2008 /

Picasso derived a posteriori error estimator in the classical L^2-norm in time and H^1-norm in space based on the energy method, and a lower bound for the local error is also derived for the associated a posteriori error indicator. Recently, Chen and Jia [7] obtained an efficient and reliable a posteriori error estimate for linear parabolic equations, which is also in the energy norm and based on a direct energy estimate argument.

In the decades, there appear many works concentrating on the adaptivity of various optimal control problems. For example, [4] studied the adaptive finite element method for the optimal control problems governed by PDE, while a posteriori error estimators for convex distributed optimal control problems governed by elliptic equations, parabolic equations, Stokes equations, integral equations and integro-differential equations are derived in [5,17,19,21–23], respectively, the a posteriori error estimates for the boundary control problems governed by elliptic equation are also discussed in [15,20].

The main objective of this paper is to establish the a posteriori error estimate of the finite element approximation for the boundary control problems governed by the parabolic partial differential equations. Three different a posteriori error estimators are provided for the parabolic boundary control problems with the observations of the distributed state, the boundary state and the final state. It is proven that these estimators are reliable bounds of the finite element approximation errors, and can be used as the indicators of the mesh refinement in adaptive finite element methods. Although we use some ideas and techniques, which have been applied in our earlier work for the parabolic distributed optimal control and the elliptic boundary control (see, e.g., [19,20,23]), in the a posteriori error estimate analysis of this paper, there are some obviously different difficulties which should be solved for the parabolic boundary control problems.

The paper is organized as follows: In section 2, we introduce the model problems and their weak formulations, provide their fully discrete finite element approximation schemes. Then we discuss the a posteriori error estimate of the finite element approximation for the parabolic boundary control problems in Section 3. We provide three different a posteriori error estimators for the parabolic boundary control problems with the observations of the distributed state, the boundary state and the final state in Subsections 3.1, 3.2 and 3.3, respectively.

2. Model Problems and Finite Element Approximations

In this section, we will introduce the boundary control problems governed by the parabolic partial differential equations with three kinds of different observations and their finite element approximations.

Let Ω be a bounded domain in $\mathbf{R}^n (n \leq 3)$ with a Lipschitz boundary $\partial\Omega$. In this paper, we adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces on Ω with norm $\|\cdot\|_{m,p,\Omega}$ and seminorm $|\cdot|_{m,p,\Omega}$. We denote $W^{m,2}(\Omega)$ by $H^m(\Omega)$ and set $H_0^1(\Omega) \equiv \{v \in H^1(\Omega) : v|_{\partial\Omega=0}\}$. We denote by $L^s(0,T;W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from $(0,T)$ into $W^{m,p}(\Omega)$ with norm

$$\|v\|_{L^s\left(0,T;W^{m,p}(\Omega)\right)} = \left(\int_0^T \|v\|_{m,p,\Omega}^s dt\right)^{\frac{1}{s}} \quad \text{for} \quad s \in [1,\infty)$$

and the standard modification for $s = \infty$. Similarly, one define the spaces $H^1\big(0,T;W^{m,p}(\Omega)\big)$ and $C^l\big(0,T;W^{m,p}(\Omega)\big)$. In addition c or C denotes a general positive constant independent of the mesh size parameter h.