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A NEW FINITE ELEMENT APPROXIMATION OF A STATE-CONSTRAINED OPTIMAL CONTROL PROBLEM*

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Abstract

In this paper, we study numerical methods for an optimal control problem with pointwise state constraints. The traditional approaches often need to deal with the delta-singularity in the dual equation, which causes many difficulties in its theoretical analysis and numerical approximation. In our new approach we reformulate the state-constrained optimal control as a constrained minimization problems only involving the state, whose optimality condition is characterized by a fourth order elliptic variational inequality. Then direct numerical algorithms (nonconforming finite element approximation) are proposed for the inequality, and error estimates of the finite element approximation are derived. Numerical experiments illustrate the effectiveness of the new approach.

Mathematics subject classification: 49J20, 65N30.
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1. Introduction

In this paper, we consider the following state-constrained optimal control problem:

$$\min_{y\le\varphi}\left\{\frac{1}{2}\int_\Omega (y-y_0)^2dx+\frac{\alpha}{2}\int_\Omega u^2dx\right\} \tag{1.1}$$

subject to

$$\begin{aligned} -\Delta y &= u \quad \text{in} \quad \Omega\\ y&=0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

where $\alpha>0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^2$ with the Lipschitz continuous boundary $\partial\Omega$, $y_0\in L^2(\Omega)$ is the desired state, and φ is a given function. We further assume that $\varphi|_{\partial\Omega}>0$, and more details will be specified later.

Such a state-constrained optimal control is a very important model in many applications and there has already existed much research on the numerical approximation of the above state constrained optimal control problem in the literature. Many numerical strategies were proposed and both a priori and a posteriori error analysis were investigated. At first, we should mention

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the work of Casas in [7], where the optimality conditions and important theoretical analysis of the problem were provided. For the standard finite element approximation of the control problem, a priori error estimates were derived by Deckelnick and Hinze in [11], where non-classic techniques were developed to handle the delta-singularity of the co-stated equation, see below. An augmented Lagrangian method was proposed to solve state and control constrained optimal control problems by Bergounioux and Kunisch in [3]. They also proposed another method: a primal-dual strategy to solve problem (1.1) in [4]. Casas proved convergence of finite element approximations to optimal control problems for semi-linear elliptic equations with finitely many state constraints in [8]. Casas and Mateos extended these results in [9] to a less regular setting for the states, and proved convergence of finite element approximations to semi-linear distributed and boundary control problems. In [25], the state-constrained control problem was approximated by a sequence of control-constrained control problems, and then the interior point method was applied to approximating the solutions. In recent years, a level set approach was applied to state-constrained problems in [16].

Furthermore all the research mentioned above was based on the first order optimality conditions of the control problem, in which an adjoint state p and a Lagrange multiplier λ are introduced. The first order optimality conditions can be stated as: The pari $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ is the unique solution of (1.1) if and only if there exist an adjoint state $p \in L^2(\Omega)$ and a Lagrange multiplier $\lambda \in \mathcal{M}(\Omega)$ such that

$$\begin{cases} -\Delta y = u \quad \text{in } \Omega, \quad y = 0 \quad \text{on } \partial\Omega \\ -(p, \Delta w)_\Omega + < \lambda, w >_{\mathcal{M},\mathcal{C}} = (y_0 - y, w)_\Omega, \quad \forall w \in H_0^1(\Omega) \cap H^2(\Omega), \\ < \lambda, z - y >_{\mathcal{M},\mathcal{C}} \leq 0, \quad \forall z \in \mathcal{C}(\Omega), \quad z \leq \varphi, \\ p = \alpha u, \\ y \leq \varphi \quad \text{in } \Omega, \end{cases} \tag{1.2}$$

where $(\cdot,\cdot)_\Omega$ denotes the L^2 inner product in Ω, $< \cdot, \cdot >_{\mathcal{M},\mathcal{C}}$ denotes the duality pairing between $\mathcal{C}(\Omega)$ and $\mathcal{M}(\Omega)$. We denote by $\mathcal{M}(\Omega)$ the space of real regular Borel measures on Ω and recall that it can be identified with the dual $\mathcal{C}^*(\Omega)$ of $\mathcal{C}(\Omega)$. In particular, every element $\lambda \in \mathcal{C}^*(\Omega)$ generates an element $[\lambda] \in \mathcal{M}(\Omega)$ such that $< \lambda, y >_{\mathcal{C}^*,\mathcal{C}} = \int_\Omega y \; d[\lambda]$ for all $y \in \mathcal{C}(\Omega)$. The details can be found in [4, 5, 16], for example.

One of the main computational difficulties in solving the above system is that the multiplier λ is often a delta measure, which has infinite values at some unknown points on the free boundary of the coincidence set $\{x : y = \varphi\}$. Whatever discretization methods are used, special care needs to be taken for these (unknown) areas in order to obtain reasonable computational efficiency. In finite element method, normally adaptive meshes are needed so that they are refined around these points guided by some error estimators. This is the main motivation of a posteriori error analysis of the finite element method. In this regard, a goal-oriented adaptive finite element concept was developed in [14], while Hoppe and Kieweg provided a posteriori error estimators of residual-type for the state constrained optimal control problem in [17]. However there seemed to still exist many issues in the formulation and analysis of these a posteriori error estimators due to the presence of the delta measure.

In this paper, we adopt a different approach for approximating this state-constrained optimal control problem, which avoids using the first order optimality conditions. The main idea is: substitute the control u in the minimizing functional (1.1) by $u = -\Delta y$, which is based on the state equation, and reformulate it as a constrained minimization problem involving only state