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Review Article

MULTIGRID METHODS FOR OBSTACLE PROBLEMS*

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Abstract

In this review, we intend to clarify the underlying ideas and the relations between various multigrid methods ranging from subset decomposition, to projected subspace decomposition and truncated multigrid. In addition, we present a novel globally convergent inexact active set method which is closely related to truncated multigrid. The numerical properties of algorithms are carefully assessed by means of a degenerate problem and a problem with a complicated coincidence set.

Mathematics subject classification: 65M55, 35J85.
Key words: Multigrid methods, Variational inequalities.

1. Introduction

Since the pioneering papers of Fichera [1] and Stampaccia [2] almost fifty years ago, variational inequalities have proved extremely useful for the mathematical description of a wide range of phenomena in material science, continuum mechanics, electrodynamics, hydrology and many others. We refer to the monographs of Baiocchi and Capelo [3], Cottle et al. [4], Duvaut and Lions [5], Glowinski [6] or Kinderlehrer and Stampaccia [7] for an introduction. Even the special case of obstacle problems covers a large and still growing number of applications ranging from contact problems in continuum mechanics to option pricing in computational finance or phase transitions in metallurgy (cf., e.g., Rodrigues [8]). In addition, the fast algebraic solution of discretized versions of highly nonlinear partial differential equations or related variational inequalities can be often traced back to a sequence of obstacle problems playing the same role as linear problems in classical Newton linearization [9–12]. Finally, apart from their practical relevance, obstacle problems are fascinating mathematical objects of their own value which inherit some, but far from all essential properties from their unconstrained counterparts.

On this background, many approaches for the iterative solution of obstacle problems have been suggested and pursued. Penalty methods based on straightforward regularization are still popular in the engineering community. A mathematically well-founded approach is to incorporate the constraints by Lagrange multipliers [6]. It is an advantage of this approach that very general constraints can be treated in a systematic way. On the other hand it doubles the number of unknowns and leads to indefinite problems. operators and box constraints. Active set strategies consist of an activation/inactivation step that produces an actual guess for the coincidence set and a subsequent solution step for the resulting reduced linear problem. This concept has been very popular since the benchmarking work by Hackbusch and Mittelmann [13] and Hoppe [14, 15]. Recent new interest was stimulated by a reinterpretation of the active set

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approach in terms of nonsmooth Newton methods [16, 17]. As the existing convergence theory typically requires the exact solution of the linear subproblems the combination with inexact (multigrid) solvers is often performed on a heuristic level [18–20].

In this review we concentrate on extensions of classical multigrid methods to self-adjoint elliptic obstacle problems with box-constraints. Our aim is to bridge the gap between the underlying simple ideas motivated by linear subspace decomposition and detailed descriptions of the final implementation as multigrid V-cycles. We also intend to clarify the relations between different concepts ranging from subset decomposition [21], projected subspace decomposition [22–24] to monotone multigrid [25] and even active set strategies both with regard to convergence analysis and numerical properties. In particular, we propose a novel truncated nonsmooth Newton multigrid method which can be as well regarded as an inexact active set algorithm or a slight modification of truncated monotone multigrid. Activation/inactivation is performed by a projected Gauß-Seidel step, linear solution is replaced by just one truncated multigrid step (cf. Kornhuber and Yserentant [26]) and global convergence is achieved by damping.

Roughly speaking, it turns out that increasing flexibility goes with decreasing theoretical coverage ranging from multigrid convergence rates for multilevel subset decomposition or projected multilevel relaxation to strong mesh-dependence of truncated monotone multigrid or truncated nonsmooth Newton multigrid for badly chosen initial iterates. On the other hand, increasing flexibility seems to increase the convergence speed considerably in the case of reasonable initial iterates: Combined with, e.g., nested iteration, truncated monotone multigrid or truncated nonsmooth Newton multigrid methods converge even for complicated coincidence sets with similar convergence speed as classical linear multigrid methods for unconstrained problems. The lack of robustness of truncated monotone multigrid or active set strategies is that local inactivation by projected Gauß-Seidel or related strategies [16] might deteriorate the convergence speed, because slow next-neighbor interaction might dominate for overestimated coincidence sets. As a natural remedy, we also propose hybrid methods where local activation/inactivation is replaced by a global standard monotone multigrid step. In our numerical experiments, hybrid version prove extremely efficient for degenerate problems.

2. Continuous Problem and Discretization

2.1. Constrained minimization, variational inequalities, and finite elements

Let Ω be a bounded, polyhedral domain in the Euclidean space $\mathbb{R}^d$, $d = 1, 2, 3$ and let $H \subset H^1(\Omega)$ be a closed subspace. We consider the minimization problem

$$u \in \mathcal{K}: \qquad \mathcal{J}(u) \leq \mathcal{J}(v) \qquad \forall v \in \mathcal{K} \tag{2.1}$$

with the closed, convex, and non-empty set $\mathcal{K}$,

$$\mathcal{K} = \{v \in H \mid v \geq \varphi \text{ a.e. in } \Omega\} \subset H,$$

as generated by a suitable obstacle function $\varphi \in H^1(\Omega) \cap C(\overline{\Omega})$. We emphasize that all algorithms and convergence results to be presented can be generalized to sets $\mathcal{K}$ where also an upper obstacle is present. The energy functional $\mathcal{J}$,

$$\mathcal{J}(v) = \tfrac{1}{2}a(v, v) - \ell(v), \tag{2.2}$$