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Review Article

FINITE ELEMENTS WITH LOCAL PROJECTION STABILIZATION FOR INCOMPRESSIBLE FLOW PROBLEMS*

Malte Braack
Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Ludewig-Meyn-Str. 4
D-24098 Kiel, Germany
Email: braack@math.uni-kiel.de
Gert Lube
Institut für Numerische und Angewandte Mathematik, Georg-August Universität Göttingen,
Lotzestrasse 16–18 D-37083 Göttingen, Germany
Email: lube@math.uni-goettingen.de

Abstract

In this paper we review recent developments in the analysis of finite element methods for incompressible flow problems with local projection stabilization (LPS). These methods preserve the favourable stability and approximation properties of classical residual-based stabilization (RBS) techniques but avoid the strong coupling of velocity and pressure in the stabilization terms. LPS-methods belong to the class of symmetric stabilization techniques and may be characterized as variational multiscale methods. In this work we summarize the most important a priori estimates of this class of stabilization schemes developed in the past 6 years. We consider the Stokes equations, the Oseen linearization and the Navier-Stokes equations. Furthermore, we apply it to optimal control problems with linear(ized) flow problems, since the symmetry of the stabilization leads to the nice feature that the operations "discretize" and "optimize" commute.

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1. Introduction

Among the most important points for the computation of flow problems is the choice of the underlying discretization in space and time. An appropriate discretization should be accurate and efficient (in terms of numerical costs). The accuracy is closely linked to their approximation and stability properties. Specifically, a finite element discretization for the Navier-Stokes equation has to deal with the stiff pressure-velocity coupling and with the advective terms for flows at higher Reynolds number.

Well established methods for steady problems are the Galerkin Least-Squares (GLS) techniques, where certain stabilization terms are added to the corresponding variational formulation. The basic idea is that those extra terms vanish for the exact (strong) solution, since they involve the strong residual. Due to this feature those methods are called *consistent*. The technique of streamline diffusion, see Johnson [1], to stabilize advective terms can also be accounted to these *residual-based* stabilization techniques.

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A drawback of the residual-based methods is that they introduce a large amount of additional couplings between the variables. This becomes troublesome especially for complex flows with compressible features or with additional variables as in reactive flow problems. Furthermore, an extension to non-steady problems is problematic since space-time finite elements have to be used. For a discussion of such problems, we refer to [2].

In this work we give a review on the *local projection stabilization* (or short LPS) which recently attracted the attention in numerical analysis. We collect the most important results for this kind of finite element stabilization and present them in an integrated framework. We mainly concentrate on the a priori results for pairs of finite elements of possibly different order for pressure and velocity. The principle idea of this symmetric stabilization technique is to project parts of the residual in order to obtain a stable scheme. Since not the full residual enters in the stabilization, the (strong) consistency is sacrificed. However, the method has a certain *weak consistency* property as explained later. As a consequence, this class of stabilization techniques is still of optimal order. Moreover, one does not need to resort to space-time finite elements for time stepping in order to stay consistent, but can apply any A-stable higher-order finite difference scheme for the discretization in time.

An outline of this paper is as follows: In Section 2, we introduce some notation and summarize basic material on the underlying finite element spaces. In Section 3, we consider the Stokes problem for the description of a viscous fluid in the domain $\Omega \subset \mathbb{R}^d$ with homogeneous boundary conditions for the velocity (no-slip):

$$-\Delta \mathbf{v} + \nabla p = f \quad \text{in } \Omega\,, \tag{1.1}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{1.2}$$
$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,. \tag{1.3}$$

Here, $p : \Omega \to \mathbb{R}$ denotes the pressure and $\vec{u} : \Omega \to \mathbb{R}^d$ the velocity field. Basic ideas of the LPS approach will be explained for equal-order finite element pairs for velocity and pressure which do not pass the well-known discrete compatibility (or inf-sup stability) condition by Babuska-Brezzi.

In Section 4, we extend the approach to the generalized Oseen system

$$-\nu\Delta \mathbf{v} + (\mathbf{b}\cdot\nabla)\mathbf{v} + \sigma\mathbf{v} + \nabla p = f \quad \text{in } \Omega\,, \tag{1.4}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{1.5}$$
$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,. \tag{1.6}$$

with constant parameters $\nu > 0$ and $\sigma \in \mathbb{R}^+$ and a given flow field $\mathbf{b} : \overline{\Omega} \to \mathbb{R}^d$. A unified analytical framework for the LPS method includes as well equal-order pairs for velocity and pressure as pairs for which the Babuska-Brezzi stability condition is valid. Moreover, the approach covers the so-called one-level and two-level variants of the LPS technique.

The Oseen system typically appears within the solution of the Navier-Stokes problem

$$\partial_t\mathbf{v} - \nu\Delta \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = f \quad \text{in } \Omega\,, \tag{1.7}$$
$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega\,, \tag{1.8}$$
$$\mathbf{v} = 0 \quad \text{on } \partial\Omega\,. \tag{1.9}$$

as auxiliary problem if an A-stable implicit time discretization is applied first. This will be considered in Section 5 as a reasonable approach to laminar flows. Moreover, the link of the