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VARIATIONAL DISCRETIZATION FOR OPTIMAL CONTROL GOVERNED BY CONVECTION DOMINATED DIFFUSION EQUATIONS*

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Abstract

In this paper, we study variational discretization for the constrained optimal control problem governed by convection dominated diffusion equations, where the state equation is approximated by the edge stabilization Galerkin method. A priori error estimates are derived for the state, the adjoint state and the control. Moreover, residual type a posteriori error estimates in the L^2-norm are obtained. Finally, two numerical experiments are presented to illustrate the theoretical results.

Mathematics subject classification: 65N30.
Key words: Constrained optimal control problem, Convection dominated diffusion equation, Edge stabilization Galerkin method, Variational discretization, A priori error estimate, A posteriori error estimate.

1. Introduction

Optimal control problem governed by convection dominated diffusion equations arises in many science and engineering applications. Recently, extensive research has been carried out on various theoretical aspects of optimal control problems governed by convection diffusion and convection dominated equations, see, e.g., [2, 3, 10, 29].

It is well known that the standard finite element discretizations applied to convection dominated diffusion problems lead to strongly oscillations when layers are not properly resolved. To stabilize this phenomenon, several well-established techniques have been proposed and analyzed, for example, the streamline diffusion finite element method [16], residual free bubbles [4], and the discontinuous Galerkin method [18]. Drawing on earlier ideas by Douglas and Dupont [11], Burman and Hansbo proposed an edge stabilization Galerkin method to approximate the convection dominated diffusion equations in [5]. The method uses least square stabilization of the gradient jumps across element edges, and can be seen as a continuous, higher order interior penalty method. The analysis of edge stabilization Galerkin methods has been extended to the Stokes equations [6], and to incompressible flow problems [7, 26].

Although above stabilization techniques are deeply studied for the convection dominated diffusion equations, their application to optimal control problems governed by convection dominated diffusion equations is not yet intensively studied. This may be due to the fact that stable numerical treatment of the optimality conditions requires stabilization for both the state and

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the adjoint equation, and it is not straightforward to chose stabilization techniques such that the approaches *first optimize, then discretize* and *first discretize, then optimize* commute. This question for example pops up if one considers the streamline upwind Galerkin method (SUPG) for discretizing the state and the adjoint equation in the optimality system, since this approach seems not to be well suited for the duality techniques frequently used in optimal control. In [3] and [29] stabilized finite element methods for optimal control governed by convection diffusion equations are applied. Both approaches use standard finite element discretization with stabilization based on symmetric penalty terms, where local projections (the so called LPS-method) are used in [3], and edge stabilization (see [5]) in [29]. Then formulating the control problem on the continuous level and then discretizing the optimality conditions appropriately is equivalent to considering the control problem on the discrete level. Hence the question posed above of which concept to apply is made redundant.

In [3] a priori error estimates are proved for both constrained and unconstrained problems, while a priori and a posteriori error estimates are provided in [29].

In [14] the first author proposes the variational discretization concept for optimal control problems with control constraints, which implicitly utilizes the first order optimality conditions and the discretization of the state and adjoint equations for the discretization of the control instead of discretizing the space of admissible controls. The application to the control governed by elliptic equations is discussed, and optimal error estimates are provided.

Here we combine variational discretization and the edge stabilization Galerkin method and apply them to the discretization of optimal control problems governed by convection diffusion equations. We first derive the continuous optimality system, which contains the state equation, the adjoint state equation and the optimality condition, which is given in terms of a variational inequality. Then similar to the standard approaches to optimal control problems governed by elliptic or parabolic partial differential equations (see, e.g., [21–24]), we derive the discrete optimal control problem by using the edge stabilization Galerkin method to approximate the state equation, whose optimality system then coincides with that obtained by discretizing the state and adjoint state in the continuous optimality system by finite elements with edge stabilization. The control is not discretized in our approach. For the control u, the state y and the adjoint state p we prove the a priori estimate

$$\| y - y_h \|_{*,\Omega} + \| p - p_h \|_{*,\Omega} + \| u - u_h \|_{0,\Omega} \leq C\Big(h^{3/2} + h\varepsilon^{1/2}\Big),$$

where u_h, y_h, p_h denote their discrete counterparts, and $\| \cdot \|_{*,\Omega}$ is defined in Section 3. We note that this result is of the same quality as those obtained in [3] and [29], but is obtained without structural assumptions like [3, Assumption 2], and also by a different simpler proof technique. Furthermore, we construct a residual type a posteriori error estimator which only contains contributions from the local residuals in the state and the adjoint equation. Contributions from the optimality condition do not appear since the control is not discretized in the variational approach taken here. Finally the numerical examples are presented to illustrate our theoretical results.

The paper is organized as follows: In Section 2, we describe the edge stabilization Galerkin scheme for the constrained optimal control problem governed by convection dominated diffusion equations using variational discretization. In Sections 3 we prove the a priori error estimate, and in Section 4, the a posteriori error estimator is constructed. In Section 5, we present two numerical examples to illustrate the theoretical results. In the last section, we briefly summarize the method used, results obtained and possible future extensions and challenges.