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A SPECTRAL METHOD FOR PANTOGRAPH-TYPE DELAY DIFFERENTIAL EQUATIONS AND ITS CONVERGENCE ANALYSIS*

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Abstract

We propose a novel numerical approach for delay differential equations with vanishing proportional delays based on spectral methods. A Legendre-collocation method is employed to obtain highly accurate numerical approximations to the exact solution. It is proved theoretically and demonstrated numerically that the proposed method converges exponentially provided that the data in the given pantograph delay differential equation are smooth.

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1. Introduction

In this paper we consider the delay differential equation:

$$u'(x) = a(x)u(qx), \quad 0 < x \leq T, \tag{1.1}$$

$$u(0) = y_0, \tag{1.2}$$

where $0 < q < 1$ is a given constant and a is a smooth function on $[0, T]$. Eq. (1.1) belongs to the class of so-called pantograph delay differential equations; see [7, 10] for details on their theory and physical applications.

The existing numerical methods for solving (1.1)-(1.2) include Runge-Kutta type methods (see, e.g., the monograph [3]) and collocation methods (cf. [1,2,4,5]). The main difficulty in the application of Runge-Kutta methods to (1.1) is the lack of information at the grid points for the function on the right-hand-side of (1.1); these numerical data have to be generated by some local interpolation process. While collocation methods yield globally defined approximations, the collocation solutions are not globally smooth. Moreover, it has been shown in [5] that for

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arbitrarily smooth solutions of (1.1) the optimal order at the grid points of collocation methods using piecewise polynomials of degree m cannot exceed $p = m + 2$ when $m \geq 2$ (in contrast to their application to ordinary differential equations where collocation at the Gauss points leads to $\mathcal{O}(h^{2m})$-convergence).

If the function a is in $C^d[0,T]$, then the corresponding solution of the initial-value problem (1.1)-(1.2) lies in $C^{d+1}[0,T]$. In this case, it is more natural to employ spectral-type methods since they produce approximate solutions that are defined globally on $[0,T]$ and are globally smooth. Moreover, the resulting errors inherit the typical property of spectral method in that they decay exponentially.

For ease of notation we will describe and analyze the spectral method on the standard interval $I := [-1,1]$. Hence, we employ the transformation

$$x = \frac{T}{2}(1+t), \quad t = \frac{2x}{T} - 1. \tag{1.3}$$

Then problem the (1.1)-(1.2) becomes

$$y'(t) = b(t)y(qt + q_1), \quad -1 < t \leq 1, \tag{1.4}$$

$$y(-1) = y_0, \tag{1.5}$$

where

$$y(t) := u\left(\frac{T}{2}(1+t)\right), \quad b(t) := \frac{T}{2}a\left(\frac{T}{2}(1+t)\right), \quad q_1 := q - 1. \tag{1.6}$$

2. The Spectral Method

Let $\{t_k\}_{k=0}^N$ be the set of the $(N+1)$-point Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto points in $[-1,1]$, and denote by $\mathcal{P}_N$ the space of polynomials with degrees not exceeding N. Integration of (1.4) from $[-1,t_j]$ gives

$$y(t_j) = y_0 + \int_{-1}^{t_j} b(s)y(qs + q_1)ds, \quad j \geq 1, \tag{2.1}$$

and the linear transformation

$$s = \frac{t_j+1}{2}v + \frac{t_j-1}{2}$$

yields

$$y(t_j) = y_0 + \int_{-1}^{1} \tilde{b}(v;t_j)\, y\left(\frac{t_j+1}{2}qv + q_{1j}\right)dv, \tag{2.2}$$

where

$$\tilde{b}(v;t_j) := \frac{1+t_j}{2}b\left(\frac{t_j+1}{2}v + \frac{t_j-1}{2}\right), \quad q_{1j} := \frac{t_j+1}{2}q - 1.$$

If we apply the $(N+1)$-point Legendre Gauss, Legendre Gauss-Radau, or Legendre Gauss-Lobatto quadrature formula to (2.2) we obtain

$$y(t_j) \approx y_0 + \sum_{k=0}^{N} \omega_k \tilde{b}(v_k;t_j)y\left(\frac{t_j+1}{2}qv_k + q_{1j}\right), \tag{2.3}$$