Journal of Computational Mathematics, Vol.27, No.2-3, 2009, 280–298.

A NUMERICAL STUDY OF UNIFORM SUPERCONVERGENCE OF LDG METHOD FOR SOLVING SINGULARLY PERTURBED PROBLEMS*

Ziqing Xie, Zuozheng Zhang and Zhimin Zhang
Key Laboratory of Computational and Stochastic Mathematics and Its Applications, Universities of Hunan Province, Hunan Normal University, Changsha 410081, China
Email: ziqingxie@hunnu.edu.cn, zuozhengzhang@yahoo.cn, zzhang@math.wayne.edu

Abstract

In this paper, we consider the local discontinuous Galerkin method (LDG) for solving singularly perturbed convection-diffusion problems in one- and two-dimensional settings. The existence and uniqueness of the LDG solutions are verified. Numerical experiments demonstrate that it seems impossible to obtain uniform superconvergence for numerical fluxes under uniform meshes. Thanks to the implementation of two-type different anisotropic meshes, i.e., the Shishkin and an improved grade meshes, the uniform $2p+1$-order superconvergence is observed numerically for both one-dimensional and two-dimensional cases.

Mathematics subject classification: 65N30.
Key words: Singularly perturbed problems, Local discontinuous Galerkin method, Numerical fluxes, Uniform superconvergence.

1. Introduction

The discontinuous Galerkin method (DGM) was first introduced in 1973 by Reed and Hill [25] for solving the neutron transport equation. Successively in 1974, Lesaint and Raviart [20] made the first analysis for the linear advection equation. Since then there has been an active development of DGM for hyperbolic, elliptic, and parabolic partial differential equations. For a fairly thorough compilation of the history of these methods and their applications see [15].

The local discontinuous Galerkin (LDG) method was first proposed by Cockburn and Shu in [11] as a generalization of the DGM proposed by Bassi and Rebay [2] for the compressible Navier-Stokes equations. In [11] the stability and error estimates for the method were studied. The first convergence analysis of the LDG method for elliptic problems was given by P. Castillo *et al.* [9]. Actually the LDG method possesses several properties which make it popular for practical computations. The LDG method is local (element-wise) conservative, a property which is particularly difficult to preserve by high-order finite elements. This method is also suitable for hp-adaptive implementation and allows a very efficient parallelization. A more detail review about the LDG method was given in [7, 15].

In recent years the numerical solutions of singularly perturbed boundary value problems have been received much attention. There are numerous papers, see, e.g., [6, 21, 23, 24, 30, 34–36], written on this subject. A book by Roos *et al.* [27] provides an extensive list of literature on this topic. One of the difficulties in numerically computing the solution of singularly perturbed problems lays in the so-called boundary layer behavior, i.e., the solution varies very rapidly in a very thin layer near the boundary.

* Received December 31, 2007 / Revised version received May 10, 2008 / Accepted June 22, 2008 /

Currently, there are mainly two ways to solve this problem. The first way is through the use of the h version on layer-adapted meshes [6, 26, 35, 36]. The uniform convergence independent of the perturbation parameter ε can be obtained when this technique is used. The second alternative is through the use of p or hp version [29, 32, 33]. The exponent rates of convergence can be established when the domain is smooth.

In [10], Celiker and Cockburn investigated the superconvergence of the numerical traces of some DG methods at the nodes of the mesh for 1-D convection-diffusion problems. Particularly, the authors proved that the superconvergence order of both numerical traces $\hat{u}_h^\varepsilon$ and $-\hat{q}_h + c\hat{u}_h^c$ is $2p+1$ when polynomials of degree at most p are used for the approximation (q_h, u_h) based on a suitably designed LDG method. Nevertheless in that paper, the uniform superconvergence of numerical traces was not investigated as the diffusion parameter ε goes to zero.

In [7], Castillo *et al.* showed, for special numerical fluxes, that the LDG method converges with the optimal rate of convergence of order h^{p+1} in the energy norm for the model problem of constant-coefficient linear convection-diffusion equation in the one spacial dimension. The first a priori error analysis of the LDG method for purely elliptic problems was given by Castillo *et al.* [9]. In this paper, meshes including elements of various shapes and general numerical fluxes were studied. It was shown that the convergence rates of the error in u and ∇u in the L^2 norm are $k + \frac{1}{2}$ and k, respectively. In [16], Cockburn *et al.*, established the superconvergence of the LDG method for multidimensional elliptic problems on Cartesion grids with special numerical fluxes. The convergence order in L^2 norm of the error in u and ∇u are $k+1$ and $k + \frac{1}{2}$, respectively, when tensor product polynomials of degree at most k are used. Comparing with the results in [9], the error bounds was improved by a factor $\sqrt{h}$. In this sense, it is a superconvergence result. Actually it is an extension to the multidimensional case of the results obtained by Castillo *et al.* [7, 8].

In [34], Xie and Zhang studied the LDG method for solving singularly perturbed convection-diffusion problems with mixed boundary condition. Their numerical test results indicate that the LDG method does not produce any oscillation outside the boundary layer region even under uniform mesh for small ε. The superconvergence rate $\mathcal{O}(h^{2p+1})$ of the numerical traces at the nodes was also proved.

In this paper we will compare two-type layer-adapted meshes when they are used in the h version of the LDG method for one and two dimensional problems. The numerical results exhibit that the LDG method does not produce any oscillation even under uniform meshes for arbitrary ε for both 1-D and 2-D cases. On the other hand, the $2p+1$ order uniform superconvergence of numerical fluxes are observed numerically for the LDG method under both the Shishkin and an improved grade meshes. Here the so-called "uniform convergence" means that the convergence rate is uniformly valid with respect to ε. It is worthwhile to point out that theoretical analysis of the uniform convergence is extremely difficult and remains an open problem for the LDG method.

The rest of this paper is organized as follows: In Section 2, the construction of the Shishkin mesh and improved grade mesh is described. The LDG method for one and two dimensions will be introduced in Section 3. In Section 4, the analysis of existence and uniqueness of the LDG solution is exhibited. We will in Section 5 present fruitful numerical results which illustrate the robustness of the LDG method for solving singularly perturbed problems based on two type layer-adapted meshes mentioned above. We end this paper with some conclusions in the final section.