Journal of Computational Mathematics
Vol.27, No.4, 2009, 484–494.

http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.27.4.015

FINITE DIFFERENCE APPROXIMATION FOR PRICING THE AMERICAN LOOKBACK OPTION *

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Abstract

In this paper we are concerned with the pricing of lookback options with American type constrains. Based on the differential linear complementary formula associated with the pricing problem, an implicit difference scheme is constructed and analyzed. We show that there exists a unique difference solution which is unconditionally stable. Using the notion of viscosity solutions, we also prove that the finite difference solution converges uniformly to the viscosity solution of the continuous problem. Furthermore, by means of the variational inequality analysis method, the $\mathcal{O}(\Delta t+\Delta x^2)$-order error estimate is derived in the discrete L_2-norm provided that the continuous problem is sufficiently regular. In addition, a numerical example is provided to illustrate the theoretical results.

Mathematics subject classification: 65M12, 65M06, 91B28.
Key words: American lookback options, Finite difference approximation, Stability and convergence, Error estimates.

1. Introduction

The pricing and hedging of derivative securities is a subject of considerable practical importance in finance. One basic type of derivative is an option which is a financial contract entered into by two parties, a buyer and a seller. The buyer of the contract obtains the right to trade an underlying asset, such as a stock, for a specified price, called the strike price, on or before a special time, called the expiration date. Options which provide the right to buy (sell) the underlying asset are known as call (put) options. When the option contract is entered into, the option buyer pays a price to the seller. In return, for the price the seller agrees to meet any obligations arising from the contract. Options which can be exercised only on the expiration date are called European, whereas options which can be exercised any time up to and including the expiration date are classified as American.

Generally speaking, the problems for pricing European options can be mathematically modelled by the well-known Black-Scholes partial differential equation [2] associated with initial/final and boundary value conditions, and some analytic solutions can be obtained for the

* Received May 23, 2008 / Revised version received August 25, 2008 / Accepted October 20, 2008 /

cases with simple payoffs. However, for most American options or European options with complex payoffs, no analytic solutions are available. Thus, the research of effective numerical methods is of considerable importance in the field of option pricing.

The objective of this article is to develop a finite difference method for pricing the lookback options with American type constrains. Lookback options, which may be European or American type, are a kind of path-dependent options whose payoffs depend on the history of the underlying asset values over some period of the options' lifetime. In recent years, many numerical methods, such as the binomial methods, finite difference and finite element methods, etc., have been proposed for pricing path-dependent options including lookback options. See, for instance, [1, 3, 7, 10, 11, 14, 15], and the references cited therein. However, for the lookback option with American style, it is still difficult to establish the convergence analysis and error estimates because of the complexity of the problems, and few results can be found in this aspect.

In this paper, we will analyze the pricing problem of lookback options on the basis of the differential linear complementary formula. First, we introduce a variable transformation to reduce the problem to an one-dimensional problem in space. Then, a fully implicit and unconditionally stable difference scheme is carefully constructed. In view of the lower regularity of the problem, we will use the notion of viscosity solution, and follow the idea and the method proposed by Barles et al. [4, 5] to give the convergence analysis. The basic principle is that any stable, consistent and monotonic discretization scheme converges. Provided that the continuous problem satisfies a strong comparison principle. Furthermore, we discuss the error estimate under the assumption that the exact solution is smooth enough. Again, we transform the discrete linear complementary problem into an equivalent variational inequality problem, which allows us to derive the error estimates more readily. The use of an implicit difference method results in a set of variational inequalities which have to be solved at each time step. We will briefly discuss the method of solving the discrete system of inequalities.

The plan of this paper is as follows. In Section 2, we review the mathematical model of pricing lookback options and establish the finite difference approximation. In Section 3, we analyze the stability and convergence for the discrete approximation problem. In Section 4, an $\mathcal{O}(\Delta t + \Delta x^2)$-order error estimate is derived in the discrete L^2-norm. Finally, in Section 5 the projected SOR method of solving the discrete system of inequalities is established, and a numerical example is provided to confirm the theoretical analysis and the efficiency of the algorithm.

2. The Finite Difference Approximation

Let $S = S(t)$ be the underlying asset price. As usual, assume that S follows the lognormal diffusion with constant volatility σ and expected return μ:

$$dS = \mu\, S\, dt + \sigma S\, dZ, \tag{2.1}$$

where $\{Z(t) : t \geq 0\}$ is a standard Brownian motion. A lookback option is a derivative product whose payoff depends on the maximum or the minimum of the realized asset price over the lifetime of the option. In this paper we will consider the lookback put option with floating strike. In this case, the payoff function can be expressed as:

$$G(t, S, J) = J - S, \quad J(t) = \max_{0 \leq \tau \leq t} S(\tau). \tag{2.2}$$