Journal of Computational Mathematics
Vol.27, No.4, 2009, 525–542.
http://www.global-sci.org/jcm
doi:10.4208/jcm.2009.27.4.017

A POSTERIORI ERROR ESTIMATE OF OPTIMAL CONTROL PROBLEM OF PDE WITH INTEGRAL CONSTRAINT FOR STATE*

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Abstract

In this paper, we study adaptive finite element discretization schemes for an optimal control problem governed by elliptic PDE with an integral constraint for the state. We derive the equivalent a posteriori error estimator for the finite element approximation, which particularly suits adaptive multi-meshes to capture different singularities of the control and the state. Numerical examples are presented to demonstrate the efficiency of a posteriori error estimator and to confirm the theoretical results.

Mathematics subject classification: 49J20, 65N30.
Key words: State-constrained optimal control problem, Adaptive finite element method, A posteriori error estimate.

1. Introduction

Finite element approximations of optimal control problems have been extensively studied in the literatures, most of which focused on control-constrained problems. In recent years, many studies have been carried out to examine finite element approximations of optimal control problems with state constraints. Many authors have studied the specific case of point-wise state constraints, where the constraints have the forms of $K = \{y : y \geqslant \varphi\}$ or $K = \{y \leqslant \varphi\}$, see, e.g., [3,6,7,9]. In engineering applications, one may care more about how to control the average value or L^2-norm of the state variable. So there exist many state-constraints of average types, such as integral constraint $K = \{y : \alpha \leqslant \int_\Omega y \leqslant \beta\}$, L^2-norm constraint $K = \{y : \int_\Omega y^2 \leqslant \gamma\}$ and so on. In this work we study a posteriori error estimates of the finite element approximation of an optimal control problem with the integral constraint for the state. It has been recently found that suitable adaptive meshes can greatly improve computational efficiency of the finite element approximation of the optimal control, see, e.g., [2,13,16–19]. Furthermore it has been observed that multi-meshes are often useful in computing optimal controls, see, e.g., [10,15]. Using different adaptive meshes for the control and the state allows to use very coarse meshes in solving the state equation and the co-state equation. Thus much computational time can be saved since one of the major computational loads in computing optimal control is to solve the state and co-state equations repeatedly.

The purpose of this work is to investigate adaptive multi-mesh finite element method for a state-constrained optimal control problem with the integral constraint. We derive an equivalent

* Received June 26, 2008 / Revised version received August 26, 2008 / Accepted October 20, 2008 /

a posteriori error estimator for this control problem and then present numerical results to confirm the effectiveness of the error estimator.

The plan of the paper is as follows. In Section 2, we construct the finite element approximation for the distributed optimal control problem with the state-constraint. In Section 3, an equivalent a posteriori error estimator is derived for the control problem. Finally numerical test results are presented in Section 4.

2. Model Problem and Its Finite Element Approximation

Let Ω be a bounded domain in $\mathbb{R}^d$, $1 \leqslant d \leqslant 3$, with the Lipschitz boundary $\partial\Omega$. Throughout the paper we use the standard notations for the Sobolev spaces, norms and seminorms. We denote the L^2-inner products in $L^2(\Omega)$ and $(L^2(\Omega))^d$ by

$$(v,w) = \int_\Omega vw, \quad \forall\ v,w \in L^2(\Omega)$$

and

$$(\mathbf{w},\mathbf{v}) = \int_\Omega \mathbf{w}\cdot\mathbf{v}, \quad \forall\ \mathbf{v},\mathbf{w} \in (L^2(\Omega))^d.$$

For a nonnegative integer m, $H^m(\Omega)$ denotes the usual Sobolev spaces and $H^1_0(\Omega) = \{v \in H^1(\Omega):\ v = 0 \text{ on } \partial\Omega\}$.

2.1. Optimal control problem and its optimality condition

Introduce function spaces $U = L^2(\Omega)$, $V = H^1_0(\Omega)$, and $W = \{v \in H^1_0(\Omega):\ \Delta v \in L^2(\Omega)\}$. Obviously, W is a Hilbert space with the norm:

$$\|v\|_W = \left(\ \|v\|^2_{H^1(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)}\ \right)^{1/2}.$$

The constraint set is defined as follows:

$$K = \Big\{v \in W:\ \int_\Omega v \geqslant \gamma\Big\}, \tag{2.1}$$

where γ is a given constant.

We will investigate the distributed convex state-constrained optimal control problem (OCP):

$$(OCP) \qquad \begin{cases} \min \quad \mathcal{J}(u,y) = \dfrac{1}{2}\displaystyle\int_\Omega (y-y_d)^2 + \dfrac{\alpha}{2}\int_\Omega u^2, \\ \quad s.t. \quad -\Delta y = u + f \ \ in\ \Omega, \quad y = 0 \ \ on\ \partial\Omega, \quad y \in K, \end{cases} \tag{2.2}$$

where $u \in U$ is the control and $y \in K$ is the state, $y_d \in L^2(\Omega)$ is the observation, $f \in L^2(\Omega)$ is a given function, and α is a given positive constant.

In [20],it is proved that the state-constrained optimal control problem (OCP) has a unique solution $(u,y) \in U \times V$. Further, the pair (u,y) is the solution of (OCP) if and only if there exists a unique pair $(p,\lambda) \in V \times \mathbb{R}^1_-$ where $\mathbb{R}^1_- = \{c \in \mathbb{R}^1;\ c \leqslant 0\}$, such that (u,y,p,λ) satisfies the following optimality conditions $(OCP\text{-}OPT)$:

$$(OCP\text{-}OPT): \qquad \begin{cases} (\nabla y, \nabla w) = (u+f, w), & \forall\ w \in H^1_0(\Omega); \\ (\nabla p, \nabla q) = (y - y_d, q) + (\lambda, q), & \forall\ q \in H^1_0(\Omega); \\ (\lambda, w - y) \leqslant 0, & \forall\ w \in K; \\ p + \alpha u = 0. \end{cases} \tag{2.3}$$