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Local Multigrid in H(curl)*

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Abstract

We consider $\boldsymbol{H}(\mathbf{curl}, \Omega)$-elliptic variational problems on bounded Lipschitz polyhedra and their finite element Galerkin discretization by means of lowest order edge elements. We assume that the underlying tetrahedral mesh has been created by successive local mesh refinement, either by local uniform refinement with hanging nodes or bisection refinement. In this setting we develop a convergence theory for the the so-called local multigrid correction scheme with hybrid smoothing. We establish that its convergence rate is uniform with respect to the number of refinement steps. The proof relies on corresponding results for local multigrid in a $H^1(\Omega)$-context along with local discrete Helmholtz-type decompositions of the edge element space.

Mathematics subject classification: 65N30, 65N55, 78A25.
Key words: Edge elements, Local multigrid, Stable multilevel splittings, Subspace correction theory, Regular decompositions of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, Helmholtz-type decompositions, Local mesh refinement.

1. Introduction

On a polyhedron $\Omega \subset \mathbb{R}^3$, scaled such that $\mathrm{diam}(\Omega) = 1$, we consider the variational problem: seek $\mathbf{u} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ such that

$$\underbrace{(\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v})_{L^2(\Omega)} + (\mathbf{u}, \mathbf{v})_{L^2(\Omega)}}_{=:\mathsf{a}(\mathbf{u},\mathbf{v})} = (\mathbf{f}, \mathbf{v})_{L^2(\Omega)} \quad \forall \mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) \ . \tag{1.1}$$

For the Hilbert space of square integrable vector fields with square integrable **curl** and vanishing tangential components on Γ_D we use the symbol $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$, see [22, Ch. 1] for details. The source term $\mathbf{f}$ in (1.1) is a vector field in $(L^2(\Omega))^3$. The left hand side of (1.1) agrees with the inner product of $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ and will be abbreviated by $\mathsf{a}(\mathbf{u}, \mathbf{v})$ ("energy inner product").

Further, Γ_D denotes the part of the boundary $\partial\Omega$ on which homogeneous Dirichlet boundary conditions in the form of vanishing tangential traces of $\mathbf{u}$ are imposed. The geometry of the Dirichlet boundary part Γ_D is supposed to be simple in the following sense: for each connected component Γ_i of Γ_D we can find an open Lipschitz domain $\Omega_i \subset \mathbb{R}^3$ such that

$$\overline{\Omega}_i \cap \overline{\Omega} = \Gamma_i \ , \quad \Omega_i \cap \Omega = \emptyset \ , \tag{1.2}$$

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Table 1.1: Important notation used in this paper

$\boldsymbol{H}(\mathbf{curl},\Omega)$	: Sobolev space of square integrable vector fields on $\Omega \subset \mathbb{R}^3$ with square integrable **curl**
$\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$	: vector fields in $\boldsymbol{H}(\mathbf{curl},\Omega)$ with vanishing tangential components on $\Gamma_D \subset \partial\Omega$
$\mathcal{M}$, $\mathcal{T}$	: tetrahedral finite element meshes, may contain hanging nodes
$\mathcal{N}(\mathcal{M})$	: set of vertices (nodes) of a mesh $\mathcal{M}$
$\mathcal{E}(\mathcal{M})$	: set of edges of a mesh $\mathcal{M}$
$\rho_K, \rho_{\mathcal{M}}$	: shape regularity measures
h	: – local meshwidth function for a finite element mesh – (as subscript) tag for finite element functions
$\mathbf{U}(\mathcal{M})$	: lowest order edge element space on $\mathcal{M}$
$\mathbf{b}_E$	: nodal basis function of $\mathbf{U}(\mathcal{M})$ associated with edge E
$V(\mathcal{M})$	: space of continuous piecewise linear functions on $\mathcal{M}$
$V_2(\mathcal{M})$	: quadratic Lagrangian finite element space on $\mathcal{M}$
$\widetilde{V}_2(\mathcal{M})$	: quadratic surplus space, see (2.19)
$b_{\boldsymbol{p}}$	: nodal basis function of $V(\mathcal{M})$ ("tent function") associated with vertex $\boldsymbol{p}$
$\mathfrak{B}_X(\mathcal{M})$	: set of nodal basis functions for finite element space X on mesh $\mathcal{M}$
$\boldsymbol{\Pi}_h$	: nodal edge interpolation operator onto $\mathbf{U}(\mathcal{M})$, see (2.7)
$\mathcal{I}_h$	: vertex based piecewise linar interpolation onto $V(\mathcal{M})$
$\mathbb{P}_p$	: space of 3-variate polynomials of total degree $\leq p$
$\overline{\mathbf{U}}(\mathcal{M})$, $\overline{V}(\mathcal{M})$	: finite element spaces oblivious of zero boundary conditions
$\prec$	: nesting of finite element meshes
$\ell(K)$	: level of element K in hierarchy of refined meshes
ω_l	: refinement zone, see (4.1)
Σ_l	: refinement strip, see (5.35)
$\mathfrak{B}^l_V$, $\mathfrak{B}^l_{\mathbf{U}}$	: sets of basis functions supported inside refinement zones, see (4.9)
Q_h	: quasi-interpolation operator for linear Lagrangian finite elements

and Ω_i and Ω_j have positive distance for $i \neq j$. Further, the interior of $\overline{\Omega} \cup \overline{\Omega_1} \cup \overline{\Omega_2} \ldots$ is expected to be a Lipschitz-domain, too (see Fig. 5.1). This is not a severe restriction, because variational problems related to (1.1) usually arise in quasi-static electromagnetic modelling, where simple geometries are common. Of course, $\Gamma_D = \emptyset$ is admitted.

Lowest order $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$-conforming edge elements are widely used for the finite element Galerkin discretization of variational problems like (1.1). Then, for a solution $\mathbf{u} \in (H^1(\Omega))^3$ with $\mathbf{curl}\,\mathbf{u} \in (H^1(\Omega))^3$ we can expect the optimal asymptotic convergence rate

$$\|\mathbf{u} - \mathbf{u}_h\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \leq C N_h^{-1/3}\,, \tag{1.3}$$

on families of finite element meshes arising from global refinement. Here, $\mathbf{u}_h$ is the finite element solution, N_h the dimension of the finite element space, and $C > 0$ does not depend on N_h. However, often $\mathbf{u}$ will fail to possess the required regularity due to singularities arising at edges/corners of $\partial\Omega$ and material interfaces [20, 21]. Fortunately, it seems to be possible to retain (1.3) by the use of adaptive local mesh refinement based on a posteriori error estimates, see [10, 47] for theory in H^1-setting, [7, 17] for numerical evidence in the case of edge element discretization, and [8, 31, 45] for related theoretical investigations.